ON SUBLINEARLY QUASISYMMETRIC HOMEOMORPHISMS

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Subriemannian Geometry and Beyond II - Jyväskylä University

Outline

This talk is about **subinearly quasisymmetric homeomorphisms** between **metric spaces**:

How do they compare to quasisymmetric homeomorphisms?

Where do they come from?

How to produce some?

Do they preserve invariants?

What are they good for?

HOW DO THEY COMPARE TO QUASISYMMETRIC HOMEOMORPHISMS?

Quasisymmetric homeomorphisms

Z, Z' are metric spaces.
$\eta : [0, +\infty) \to [0, +\infty)$ is an increasing homeomorphism.
A homeomorphism $f : Z \to Z'$ is **quasisymmetric** if for any distinct x, y, z in X and $t > 0$,

$$d(x, y) \leqslant td(x, z) \implies d(f(x), f(y)) \leqslant \eta(t)d(f(x), f(z)).$$

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Tukia-Väisälä: if Z, Z' are uniformly perfect (e.g. **connected** or Cantor) one may assume $\eta(t) = \sup\{t^{\alpha}, t^{1/\alpha}\}$ and quasisymmetric homeomorphisms are **biHölder continuous** on bounded subspaces.

Rings and asphericities

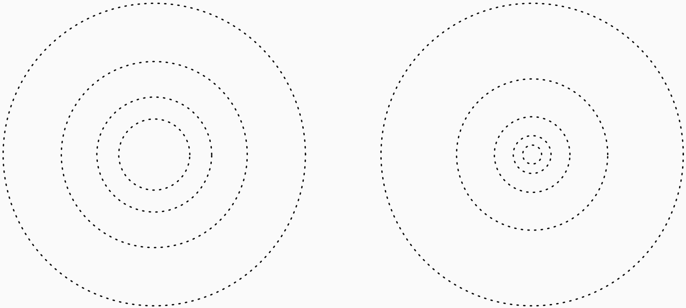
Let $t \geqslant 1$. A pair of subsets (a, a^+) of a metric space is a **t-ring** if there is a ball B such that $B \subseteq a \subseteq a^+ \subseteq tB$. radius($B$) is an **inner radius** and $\tau = \log t$ is called an **asphericity** for (a, a^+). If $a = a^+$, **round set**.

A η-quasisymmetric homeomorphism sends t-rings to $\eta(t)$-rings: it preserves bounded asphericity.

Definition

Let $\sigma \in (0, 1)$. A family of rings (a_n, a_n^+) with inner radii σ^n and asphericities τ_n is said to have **sublinear asphericity** if $\tau_n \ll n$. A homeomorphism is sublinearly quasisymmetric if it is biHölder continuous and **preserves sublinear asphericity**.

Sublinear asphericity is preserved



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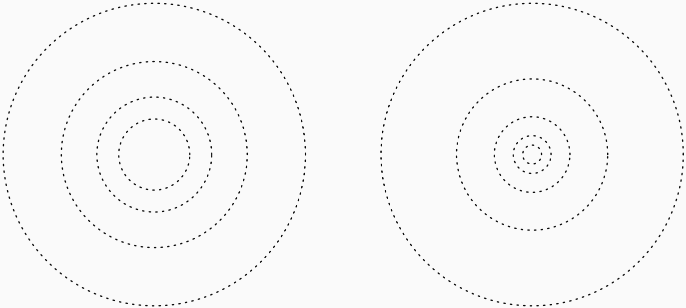

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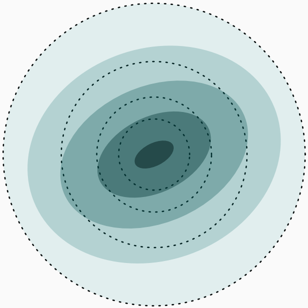

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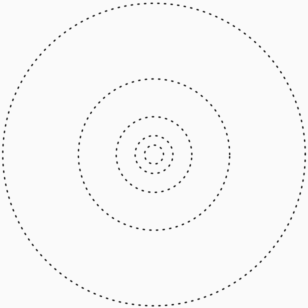

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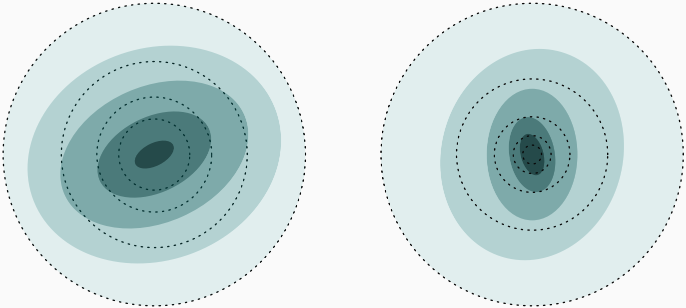

$\tau_n = O(\sqrt{n})$ $\qquad$ $\tau'_n = O(\sqrt{n})$

Figure: Round sets and their images in Euclidean $\mathbf{R}^2$.

WHERE DO THEY COME FROM?

Hyperbolic cone and Gromov boundary

The quasisymmetry class of Z can be considered a **Gromov boundary** of a **large-scale structure** or hyperbolic cone $Y = \mathrm{Con}(Z)$.

$Z = \partial_\infty Y$	$Y = \mathrm{Con}(Z)$
Euclidean $\mathbf{R}^n$	$\mathbb{H}^{n+1} = \{\text{scalar dilation group}\} \ltimes \mathbf{R}^n$
Subriemannian $\mathbf{Heis}^n$	$\mathbb{H}_{\mathbf{C}}^{n+1} = \{\text{Carnot dilation group}\} \ltimes \mathbf{Heis}^n$
Unipotent $\mathbf{R}^2$	$\{\text{unipotent dilation group}\} \ltimes \mathbf{R}^2$
Diagonal $\mathbf{R}^2$	$\{\text{diagonal dilation group}\} \ltimes \mathbf{R}^2$
q.s. homeo $Z \to Z'$	**quasiisometry** $Y \to Y'$

Sublinearly quasisymmetric homeomorphisms $Z \to Z'$ are boundary maps of **sublinearly biLipschitz equivalences** $Y \to Y'$, that are isomorphisms of the **sublinear large-scale structures** arising from work of Cornulier and Dranishnikov-Smith.

Self-similar (nongeodesic) planes

Space	Normed $\mathbf{R}^2$	Unipotent $\mathbf{R}^2$	Diagonal $\mathbf{R}^2$ ($\mu > 1$)
Dilations	$\left\{\begin{pmatrix} e^\tau & 0 \\ 0 & e^\tau \end{pmatrix}\right\}$	$\left\{\begin{pmatrix} e^\tau & \tau e^\tau \\ 0 & e^\tau \end{pmatrix}\right\}$	$\left\{\begin{pmatrix} e^\tau & 0 \\ 0 & e^{\mu\tau} \end{pmatrix}\right\}$

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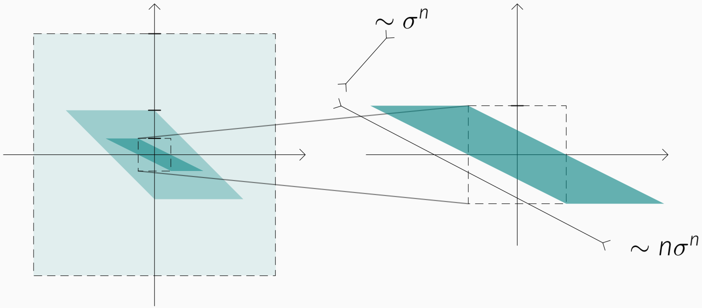

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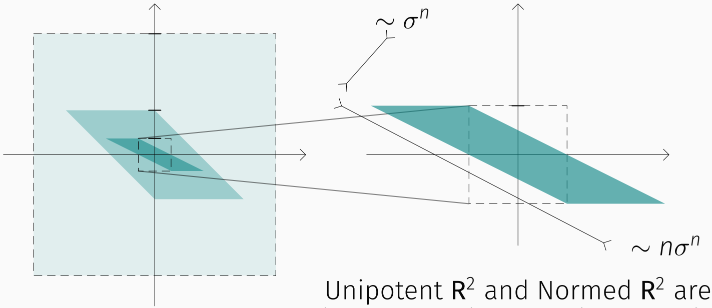

Unipotent $\mathbf{R}^2$ and Normed $\mathbf{R}^2$ are sublinearly quasisymmetric, $\tau_n = O(\log n)$

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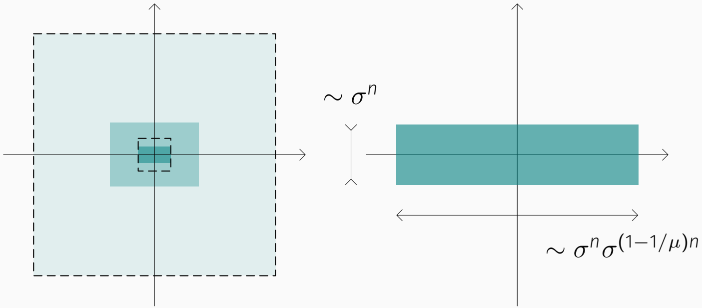

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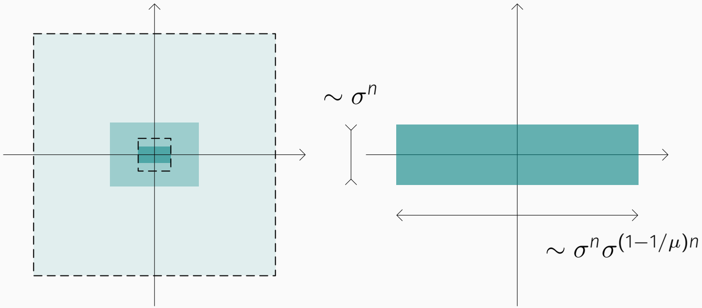

HOW TO PRODUCE SOME?

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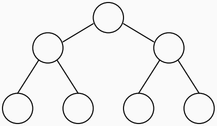

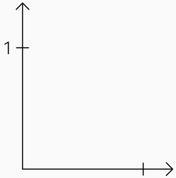

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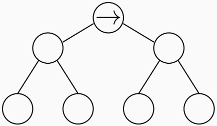

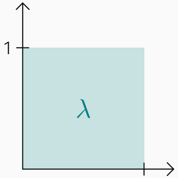

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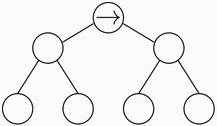

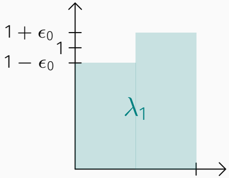

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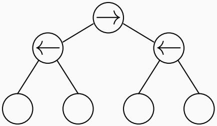

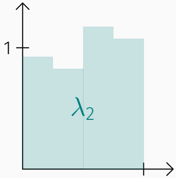

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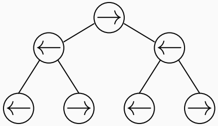

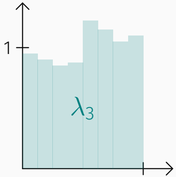

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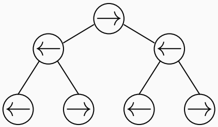
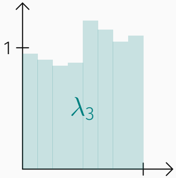


$$M = \lim_n \lambda_n$$



A method to produce sublin-q.s. homeos (II)

2nd step: Take the primitive $\phi : [0,1] \to [0,1]$ in the distributional sense.

- ϕ is not absolutely continuous. The derivative is λ-a.e. 0. The modulus of continuity deviates sublinearly from that of a Lipschitz function: $\log|\phi(x) - \phi(y)| \leqslant \log|x-y| + v(\log|x-y|)$, sublinear v.

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- **Proposition**

  ϕ and Φ are sublinearly quasisymmetric. The asphericity distorsions at scale s for ϕ and Φ are bounded by $(\sum_{n<\log_2 s} \epsilon_n)$ (in fact they are a.e. much lower).

DO THEY PRESERVE INVARIANTS?

Dimensions

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$\{\tau_j\}$ are asphericity parameters for Caratheodory/packing measures. Can be sublinearized and **sometimes computed** using the diffusivity lemma from conf. dim theory.

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Space	Sublinear conformal dimension
Normed $\mathbf{R}^2$	2
Diagonal $\mathbf{R}^2$	$1 + \mu$
Carnot group with CC metric d	$\mathrm{Hdim}(d)$
Self-similar (nongeodesic) nilpotent	**trace** of the generator of dilations.

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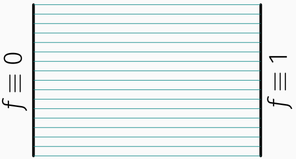

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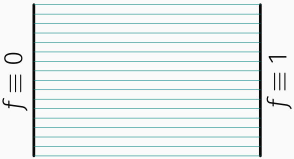

$E_p^{\{\tau_j\}}(f) \geqslant \mathrm{mod}_p^{\{\tau_j\}}(\Gamma)$

One can define **functions of locally bounded** p**-energy** $\mathcal{W}^p$; if φ is a sublin-q.s. homeo then $\mathcal{W}^p(\Omega) \xrightarrow{\sim} \mathcal{W}^p(\varphi^{-1}\Omega)$ for Ω an open in the target. $\mathcal{W}^p(\Omega)$ is a Fréchet algebra whose **spectrum** is a quotient of Ω, the largest space of leaves that it separates.

WHAT ARE THEY GOOD FOR?

Metric geometry of neg. curved 3-dim Lie groups

Metric classifications of Lie groups (with left invariant Riemannian metrics): quasiisometry, sublinear biLipschitz equivalence, may be made isometric.

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Two three dimensional (solvable) negatively curved Lie groups are quasiisometric if and only if they can be made isometric.

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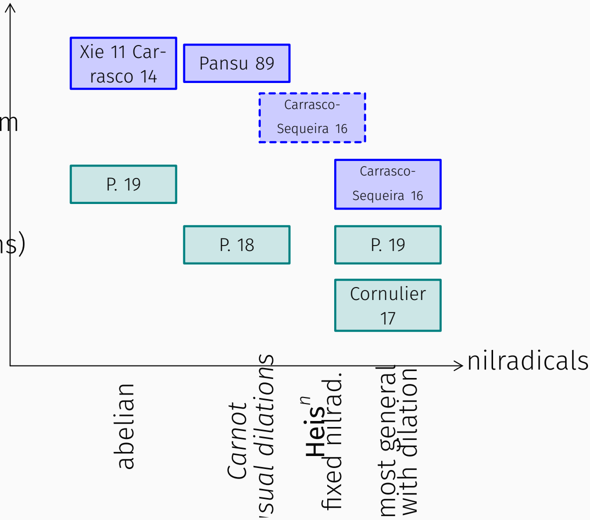
Method 1. Sublinear conformal dimension $\rightsquigarrow$ **trace**

HIGHER DIM NEG. CURVED LIE GROUPS: OVERVIEW



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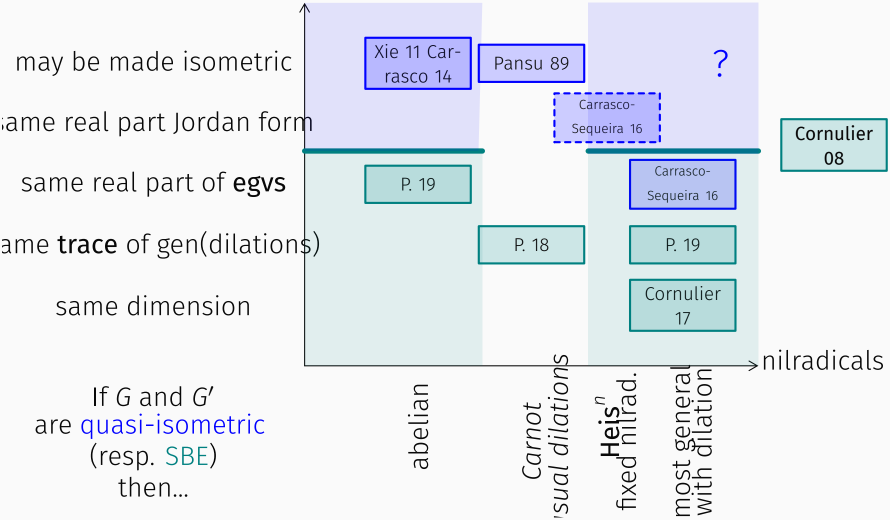

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