Free Seminar SMA.03803 / SMA.03805
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Below is a list of themes for the free seminar, autum semester, revolving around graphs and geometry. It is strongly advised to read and follow at least partially the references suggested; however other references are also welcome, if properly acknowledged.

Please contact me for the preparation of the seminar

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1. **Cauchy's rigidity theorem and the flexahedra** Two convex polyhedra P and P' are said to be *combinatorially equivalent* if there are three bijections between their set of vertices, edges and faces, that respect the incidence relations. The goal of this seminar is to give a proof of Cauchy's rigidity theorem: if two polyhedra P and Q are combinatorially equivalent and with congruent faces, then they are congruent. One may then discuss on the hypotheses, and include a treatment of a version of Euler's formula. *Reference:* [1, Chapters 12 and 13] (One may also consult [3, Chapter 2] and [5] regarding Euler's formula).

2. **Dehn's invariant and Hilbert's third problem** If two triangles in the plane have equal area, it happens that they can be dissected and recomposed into one another from smaller triangles. The analogous is false for tetrahedra, as proved by Dehn who exhibited a necessary and sufficient condition. The goal of this seminar is to give a proof of the necessary condition, using the "cone Lemma" from [1]. *Reference*: [1, Chapter 9].

3. **Projective geometry and friendship graphs.** The friendship theorem states that if in a finite graph, any two vertices have exactly one common neighbor, then the diameter of the graph is 2 and one vertex is the neighbor of any other. The goal of this seminar is to discuss a proof of the friendship theorem based on finite projective planes. *References:* One could either

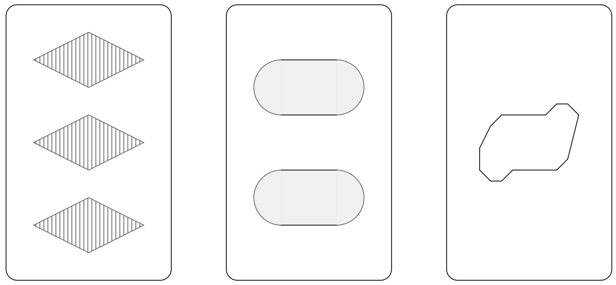

Figure 1: A set of cards.

follow [7] where the end of the proof uses [1, Chapter 39], or (with more preparatory theoretical work but a shorter proof) use [7] in combination with [2].

$3\frac{1}{2}$. **From planar graphs to the Sylvester-Gallai theorem** The Sylvester-Gallai theorem reads as follows: *Given any set of $n \geqslant 3$ points in the plane, either all the points lie on a common line or there is a line that contains exactly two points.* Somehow surprisingly in view of its formulation, a very short proof of this theorem uses the Euclidean distance structure on the plane, and the notion of "betweenness" for points on the real line. The goal of this seminar is to discuss another proof, using Euler's formula and its application [1, Proposition (B) p.77] . Other applications can be given, e.g. the one on monochromatic lines [1, p.79]. *References:* [1, Chapter 12]

$3\frac{2}{3}$. **How many edges can some graphs have?** This seminar is concerned with *extremal combinatorics*, more precisely with the following question:

> Given a finite graph H and a (large) integer n, what is the maximal number of edges that a graph on n vertices can have, provided that it does not have a subgraph isomorphic to H?

A first result in this direction is the following, where we recall that a p-clique in a finite graph is a complete subgraph on p vertices.
Turan's theorem *Let $p \geqslant 2$. A finite graph G without p-clique has n vertices and m edges, then*

$$m \leqslant \frac{p-2}{p-1}\frac{n^2}{2}.$$

The first goal of this seminar is to prove Turán's theorem using your favorite proof from [1, Chapter 36]. After this results, Erdös and Stone proved a generatlisation in which the complete rgaphs are replaced with multipartite ones. Some applications available on request. *Reference:* [1]

4. **The polynomial method and large subsets with no arithmetic progressions.**

   The card game SET is made with 81 cards, all presenting one, two or three patterns with varying features: shape, color, filling style. The goal of this game is to spot triples of cards whose features all coincide or differ (e.g. three cards with two patterns each, of the same shape, different fillings and different colors). How many cards should one draw to be sure to find such a triple? Let $\mathbf{F}_3$ denote the field with 3 elements. A higher-dimensional analogue (and more mathematically phrased) version of the previous question is the following: What is the size of the largest subset of $\mathbf{F}_3^n$ containing no arithmetic progression? This problem has seen considerable progress thanks to the introduction of a recent, elementary method in additive combinatorics. The goal of this seminar is to give an introduction to this method, and an application which could be the one given in [8]. *References:* [4], [8]

5. **Constructing Ramanujan graphs** The *girth* $g(X)$ of a simple graph X is the length of its smallest closed cycle. Let $k \geqslant 3$ be an integer. If a family of k-regular[1] finite graphs (X_n) whose number of vertices tends to $+\infty$, for all ε greater than 0, one can show that

   $$g(X_n) \leqslant (2+\varepsilon)\log_{k-1}|X_n|$$

   for n large enough. The family (X_n) of k-regular graphs is called a family of graphs with large girth if conversely, $g(X_n) \geqslant c\log|X_n|$ for some positive c. The mere existence of such objects is not obvious! The goal of this seminar is to outline a construction of such families. *Reference:* [9] Warning: this seminar has slightly more prerequisite than the previous ones. Please consult the reference before picking this subject.

References

[1] M. Aigner and G.M. Ziegler, *Proofs from the Book*, 4th edition, Springer 2010.

[2] R. Baer, Polarities in finite projective planes. *Bull. Amer. Math. Soc.* 52 (1946), 77–93.

[3] H.S.M. Coxeter, *Regular polytopes*, Dover, 1973.

[4] J. Ellenberg and D. Gijswijt, On large subsets of $\mathbf{F}_q^n$ with no three-term arithmetic progression, *Ann. Math* vol 185 (2017) `https://annals.math.princeton.edu/2017/185-1/p08`

[1] A graph is k-regular if every vertex has exactly k neighbor

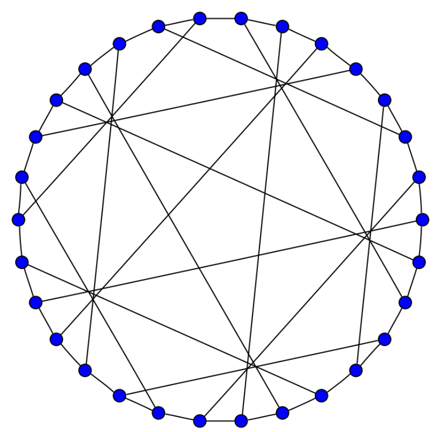

Figure 2: The Coxeter-Tutte graph is 3-regular; it has girth 8 and 30 vertices.

[5] I. Lakatos, *Proofs and refutations. The logic of mathematical discovery.* Cambridge University Press, Cambridge, 2015.

[6] J. Meier, *Groups, Graphs and Trees* London Mathematical Society Student Texts **73**.

[7] G. Pallier, *Une géométrie pour les graphes d'amitié*, Quadrature, 2016. https://www.imo.universite-paris-saclay.fr/~pallier/pdfs/ami.pdf

[8] G. Pallier, *SET: La pêche à la ligne*, Quadrature, 2017. https://www.imo.universite-paris-saclay.fr/~pallier/pdfs/reset.pdf

[9] A. Valette, *Graphes à grand tour de taille* http://images.math.cnrs.fr/pdf2004/Valette.pdf