LARGE-SCALE SUBLINEARLY LIPSCHITZ GEOMETRY OF HYPERBOLIC SPACES

GABRIEL PALLIER

Abstract. Large-scale sublinearly Lipschitz maps have been introduced by Yves Cornulier in order to precisely state his theorems about asymptotic cones of Lie groups. In particular, Sublinearly biLipschitz Equivalences (SBE) are a weak variant of quasiisometries, with the only requirement of still inducing biLipschitz maps at the level of asymptotic cones. We focus here on hyperbolic metric spaces and study properties of boundary extensions of SBEs, reminiscent of quasiMöbius (or quasisymmetric) mappings. We give a dimensional invariant of the boundary that allows to distinguish hyperbolic symmetric spaces up to SBE, answering a question of Druţu.

Introduction

0.A. **Main definitions and results.** Sublinearly Lipschitz maps between metric spaces have been gradually made into an object of study by Y. Cornulier in a series of papers starting in 2008 [7, 8, 11]. Here is a short definition of a sublinearly biLipschitz equivalence (compare to Definition 1.4):

Definition 0.1. Let X and Y be pointed metric spaces. In X and Y, denote the distances by $| \cdot - \cdot |$ and distances to the base-point by $| \cdot |$. A map $f : X \to Y$ is called a sublinearly biLipschitz equivalence (SBE) if there exists a nondecreasing, doubling function $u : \mathbf{R}_{\geqslant 0} \to \mathbf{R}_{\geqslant 1}$ with $u(r) \ll r$ as $r \to +\infty$, and $(\underline{\lambda}, \overline{\lambda}) \in \mathbf{R}^2_{>0}$ such that for any x, x' in X and y in Y,

$$\underline{\lambda}|x - x'| - u(|x| \vee |x'|) \leqslant |f(x) - f(x')| \leqslant \overline{\lambda}|x - x'| + u(|x| \vee |x'|), \tag{0.1}$$

$$\inf \{|y - y'| : y' \in f(X)\} \leqslant u(|y|), \tag{0.2}$$

where $|x| \vee |x'|$ denotes $\sup \{|x|, |x'|\}$.

Note that while the function u in the definition may depend up to an additive (or multiplicative, as u takes values higher than 1) error on base-points, the large-scale Lipschitz and reverse Lipschitz data $(\underline{\lambda}, \overline{\lambda})$ do not. The technical conditions on u are required so that there is a well-behaved notion of $(\lambda, O(u))$-sublinearly biLipschitz equivalence (resp. $(\lambda, o(u))$-sublinearly biLipschitz equivalence) between non-pointed metric spaces; it is useful to retain only the class $O(u)$ or $o(u)$ for composition purposes, see Cornulier [11, Proposition 2.2] and section 1 below. When

Date: December 30, 2018.
2010 *Mathematics Subject Classification.* Primary 20F67, 30C65; Secondary 53C23, 20F69, 30L10.
Key words and phrases. hyperbolic groups, quasiconformal mappings, cross-ratios, asymptotic geometry, analysis on metric spaces.
The author thanks the support of project ANR-15-CE40-0018.

$u = 1$, $O(u)$-sublinearly biLipschitz equivalences are the more traditional quasiisometric maps.

Sublinearly Lipschitz maps were devised in the first place so that for any nonprincipal ultrafilter ω over $\mathbf{Z}_{\geqslant 0}$ or $\mathbf{R}_{\geqslant 0}$ and scaling sequence (λ_j), $\mathrm{Con}_\omega(\cdot, \lambda_j)$ (with fixed basepoint) defines a functor from the large-scale sublinearly Lipschitz category to the Lipschitz category [8, Proposition 2.9]. The asymptotic cone characterization of hyperbolicity (Gromov [21, 2.A], Druţu [13, 3.A.1.(iii)]) ensures that within the class of quasihomogeneous, geodesic metric spaces (such as finitely generated groups), hyperbolicity is preserved by sublinearly biLipschitz equivalences (see Cornulier [11, Proposition 4.2]). However, while asymptotic cones up to biLipschitz homeomorphisms are fine SBE invariants in order to distinguish, e.g., nilpotent groups, this is not the case in the hyperbolic setting, since all complete nonpositively curved Riemannian manifolds and nonelementary Gromov-hyperbolic groups share the same asymptotic cones, namely the universal $2^{\aleph_0}$-branched $\mathbf{R}$-tree, even defined up to isometry (see for instance Erschler and Polterovich [15, Theorem 1.1.3]). This suggests to study the effects of SBEs on other asymptotic invariants instead. In this direction, Cornulier proved that sublinearly biLipschitz equivalences induce biHölder homeomorphisms between geodesic boundaries of proper geodesic hyperbolic metric spaces equipped with visual distances [11, Theorem 1.7 and Theorem 4.3]. Restated within the spaces, this says that for pairs of triples of far apart points sent to each other by a sublinearly biLipschitz equivalence, Gromov products in the source and target are within linear control of each other, a feature which may be derived from the large scale biLipschitz behavior. Similarly to Gromov products, cross-differences, or positive logarithms of cross-ratios, have an incarnation as large distances within the space, so that one can hope that the same control remains between them, with a sublinear error term. This is our main result.

Theorem 1 (Restatement of Theorem 4.2)**.** *Let $f : X \to Y$ be a $(\underline{\lambda}, \overline{\lambda}, O(u))$-sublinearly biLipschitz equivalence between hyperbolic proper geodesic spaces. Then f induces a map φ between the geodesic boundaries with the property that for all distinct $(\xi_1, \dots \xi_4)$ on the geodesic boundary of X, all of them close enough,*

$$\underline{\lambda} \log^+[\xi_i] - v\left(\overline{\boxtimes}\{\xi_i\}\right) \leqslant \log^+[\varphi(\xi_i)] \leqslant \overline{\lambda} \log^+[\xi_i] + v\left(\overline{\boxtimes}\{\xi_i\}\right), \tag{0.3}$$

where $v = O(u)$ is a sublinear function, $\log^+(s) = \max(0, \log s)$ for all $s \in \mathbf{R}_{>0}$, $\overline{\boxtimes}\{\xi_i\}$ denotes the supremum of all Gromov products over pairs in the four ξ_i's, and the brackets $[\xi_i]$ denote the cross-ratios $[\xi_1, \dots \xi_4]$ (see 1.C for definitions).

The homeomorphisms as in (0.3) are given the name of sublinearly quasiMöbius (Definition 4.1). A distinctive feature of sublinearly quasiMöbius homeomorphisms is that their distortion of the moduli of small annuli (or "eccentricity" of small ellipsoids) is bounded at small, non-infinitesimal scale:

Definition 0.2. Let Ξ be a metric space. An annulus A of Ξ is a difference of concentric balls $B(\xi, s) \setminus B(\xi, r)$ for some $\xi \in \Xi$ and $r, s \in \mathbf{R}_{>0}$. The real number $\mathfrak{M} = \log(s/r)$ is called a modulus[1] for A.

Proposition 0.3 (Restatement of Proposition 4.9)**.** *Let Ξ and Ψ be compact, uniformly perfect metric spaces and $\varphi : \Xi \to \Psi$ a $(\lambda^{-1}, \lambda, O(u))$-sublinearly quasiMöbius*

[1]This would be an ill-defined function if applied to the set A since ξ, r, s may vary, nevertheless we write that A is an annulus of modulus $\mathfrak{M}$. It mostly matters to bound moduli from above.

homeomorphism. Let A be an annulus of inner radius r, outer radius R and modulus $\mathfrak{M}$. There exists $w = O(u)$ such that if R is sufficiently small, $\varphi(A)$ is contained in an annulus of modulus

$$\mathfrak{M}' = 2\lambda\mathfrak{M} + w(-\log r).$$

When $u = 1$ this is a characterization of power-quasisymmetric mappings, compare Mackay and Tyson, [23, Lemma 1.2.18]. With their scale-sensitive moduli distortion, sublinearly quasiMöbius homeomorphisms may lack the analytic properties of quasisymmetric mappings, even between Euclidean spaces. Nevertheless we prove that they preserve the Hausdorff dimension of visual metrics in a favorable setting:

Proposition 0.4 (Consequence of Proposition 5.9)**.** *Let Ξ^* and Ψ^* be punctured*[2] *boundaries of purely real, normalized Heintze groups of Carnot type with homogeneous dimensions p and p' (see 5.B for definitions). Assume there exists a homeomorphism $\varphi : \Xi^* \to \Psi^*$ which is sublinearly quasiMöbius over any compact subset (with respect to the visual metrics). Then $p = p'$.*

The Heintze groups of Carnot type form an intermediate class between hyperbolic symmetric spaces and simply connected negatively curved homogeneous spaces. The invariance of the topological dimension of the geodesic boundary is more generally granted by Cornulier's theorem on biHölder continuity. Once combined, those two asymptotic invariants allow to distinguish all hyperbolic symmetric spaces, answering a question of Druţu [11, Question 1.16 (2)]:

Theorem 2. *Let X and Y be rank one Riemannian symmetric spaces of noncompact type. If there exists a sublinearly biLipschitz equivalence between X and Y, then X and Y are homothetic.*

In view of the Cornulier-Tessera characterization of hyperbolic connected Lie groups [12, Corollary 3], this can be rephrased as: if there exists a large-scale sublinearly biLipschitz equivalence between two hyperbolic, non-amenable connected real Lie groups G and G', then G and G' are commable.

0.B. **Acknowledgements.** Yves Cornulier suggested to look for the main theorem, my advisor Pierre Pansu provided constant help and section 5 owes a lot to his ideas. I benefited from a conversation with Peter Haïssinsky about Lipschitz-Möbius constants, explanations by Prof. Viktor Schroeder on cross-ratios, translation by Aliaksandr Minets on parts of [16], and a key remark of Sébastien Gouëzel on the proof of the quantitative Morse lemma, see 2.C herein. I thank the anonymous referee for accurate and useful comments.

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[2] The puncture is made at a distinguished point so that the remaining part of the boundary is transitively acted upon by the group; see 5.B.

1. Background

1.A. **Large-scale Sublinearly Lipschitz maps.** Here is a summary of Cornulier's definitions included for the reader's convenience. Call admissible any function $u : \mathbf{R}_{\geqslant 0} \to \mathbf{R}_{\geqslant 1}$ with the following properties:

(1) u is nondecreasing
(2) u is doubling: $\limsup_{r \to +\infty} u(2r)/u(r) < +\infty$
(3) u is strictly sublinear: $\lim_{r \to +\infty} u(r)/r = 0$.

It is not really restrictive, and in fact useful in statements, to allow such a function to be only eventually defined and conditions (1), (2) to hold only on a neighborhood of $+\infty$ in $\mathbf{R}_{\geqslant 0}$. However we will frequently work with a precise admissible function u while keeping track on explicit bounds, and where they become valid. To facilitate this we introduce the following notation:

- For all $\varepsilon > 0$, $r_\varepsilon(u)$ is $\sup\{r \in \mathbf{R}_{\geqslant 0} : u(r) > \varepsilon r\}$. This is finite by (3).
- Properties (1), (2) and the fact that $\inf_r u(r) > 0$ ensure that for any $\tau > 1$, $\sup_r u(\tau r)/u(r)$ is finite. We shall denote this number $u \uparrow \tau$.

The following lemma is for our use only; it describes the way in which the constants $r_\varepsilon(u)$ and $u \uparrow \tau$ evolve when shifting function u.

Lemma 1.1. *Let u be an admissible function. For any $p \in \mathbf{R}_{>0}$, define $u_p : \mathbf{R}_{\geqslant 0} \to \mathbf{R}_{\geqslant 1}$ as $u_p(t) = u(p+t)$. Then*

(1) *for all $\tau \in \mathbf{R}_{>1}$, $u_p \uparrow \tau \leqslant u \uparrow \tau$.*
(2) *For all $\varepsilon \in \mathbf{R}_{>0}$, if $p \geqslant r_{\varepsilon/2}(u)$ then*

$$r_\varepsilon(u_p) \leqslant \frac{u \uparrow 2}{\varepsilon} u(p).$$

Proof. Start with (1). By definition, u is nondecreasing, hence

$$u_p \uparrow \tau = \sup_r \frac{u(\tau r + p)}{u(r+p)} \leqslant \sup_r \frac{u(\tau r + \tau p)}{u(r+p)} = u \uparrow \tau.$$

As for (2), the hypothesis made on p means that for all p' greater than p, $u_p(p') \leqslant \frac{\varepsilon}{2}(p+p') \leqslant \varepsilon p'$, so $r_\varepsilon(u_p) \leqslant p$, and then $p + r_\varepsilon(u_p) \leqslant 2p$. Also note that since u_p is nondecreasing, $\varepsilon r_\varepsilon(u_p)$ is equal to $u(p + r_\varepsilon(u_p))$, so that

$$\varepsilon r_\varepsilon(u_p) = u(p + r_\varepsilon(u_p)) \leqslant u(2p) \leqslant (u \uparrow 2) u(p).$$

Finally $r_\varepsilon(u_p) \leqslant \varepsilon^{-1}(u \uparrow 2) u(p)$. □

In the following, let u be an admissible function, and let X and Y be two pointed metric spaces. Recall that whenever r and s are real numbers, $r \vee s$ denotes $\sup\{r, s\}$ and $r \wedge s$ denotes $\inf\{r, s\}$.

Definition 1.2. A map $f : X \to Y$ is called $(\overline{\lambda}, O(u))$-Lipschitz if there exists $\overline{\lambda} \in \mathbf{R}_{>0}$ (called a large-scale Lipschitz constant) and a nondecreasing function $v = O(u)$ such that for all $(x_1, x_2) \in X^2$,

$$|f(x_1) - f(x_2)| \leqslant \overline{\lambda} |x_1 - x_2| + v(|x_1| \vee |x_2|).$$

We may write that f is $(\overline{\lambda}, v)$-Lipschitz to put emphasis on v, or on the contrary a $O(u)$-Lipschitz map if the actual Lipschitz constant and function v are not relevant.

Definition 1.3. $f, g : X \to Y$ are $O(u)$-close if $|f(x) - g(x)| = O(u(|x|))$.

One checks that $O(u)$-Lipschitz maps can be composed (with a multiplicative effect on large-scale Lipschitz constants), in a way compatible with $O(u)$-closeness [11, Proposition 2.2], hence there is a well-defined category $\mathcal{L}_{O(u)}$ with metric spaces as objects[3] and large-scale $O(u)$-Lipschitz maps modulo $O(u)$-closeness as morphisms.

Definition 1.4 (compare Definition 0.1)**.** $f : X \to Y$ is a $O(u)$-Sublinearly Bilipschitz Equivalence (SBE) if the $O(u)$-closeness class of f is an isomorphism in $\mathcal{L}_{O(u)}$. This can be metric-geometrically rephrased as follows [11, Proposition 2.4]:

(1) f is $O(u)$-Lipschitz;
(2) f is $O(u)$-expansive : there exists a nondecreasing $v = O(u)$ and $\underline{\lambda} \in \mathbf{R}_{>0}$ such that
$$\forall (x_1, x_2) \in X^2,\ |f(x_1) - f(x_2)| \geqslant \underline{\lambda}|x_1 - x_2| - v(|x| \vee |x'|);$$
(3) f is $O(u)$-surjective : for $y \in Y$,
$$d(y, f(X)) = O(u(|y|)).$$

Conditions (1) and (2) alone define the notion of a $O(u)$-Lipschitz embedding ; precisely a $(\underline{\lambda}, \overline{\lambda}, v)$-embedding is a map such that
$$\forall (x_1, x_2) \in X^2,\ \underline{\lambda}|x_1 - x_2| - v(|x| \vee |x'|) \leqslant |f(x_1) - f(x_2)| \leqslant \overline{\lambda}|x_1 - x_2| + v(|x_1| \vee |x_2|).$$
We will give an equivalent definition in subsection 3.A. If there exists an admissible u such that f is a $O(u)$-sublinearly biLipschitz equivalence (resp. embedding), then f is called a sublinearly biLipschitz equivalence (resp. embedding). In some occasion, we will abbreviate $(\underline{\lambda}, \overline{\lambda})$ into a single biLipschitz constant $\lambda = \sup\{\overline{\lambda}, 1/\underline{\lambda}\}$ and call f a $(\lambda, O(u))$-sublinearly biLipschitz equivalence.

Two metric spaces X and Y such that there exists a sublinearly biLipschitz equivalence $f : X \to Y$ are called asymptotically biLipschitz in Druţu and Kapovich's book [14, 10.8].

1.B. **Gromov products and Cornulier's estimates.** Let X be a metric space. Recall that for $x_0, x_1, x_2 \in X$, the Gromov product of x_2 and x_3 seen from x_0 is by definition $(x_2 \mid x_3)_{x_0} := \frac{1}{2}(|x_1 - x_0| + |x_2 - x_0| - |x_1 - x_2|)$, and that X is δ-hyperbolic (as defined by M.Gromov [20, 1.1.C]) if there exists $\delta \in \mathbf{R}_{\geqslant 0}$ such that

$$\forall (x_0, x_1, x_2, x_3) \in X^4,\ (x_1 \mid x_3)_{x_0} \geqslant \inf\{(x_1 \mid x_2)_{x_0}, (x_2 \mid x_3)_{x_0}\} - \delta. \tag{1.1}$$

If X is δ-hyperbolic and geodesic, then in addition, the Rips inequality is available: triangles in X are 4δ-slim, [19, 2.21]. A Cauchy-Gromov sequence in X is a sequence $(x_n)_{n \in \mathbf{Z}_{\geqslant 0}}$ such that $(x_n \mid x_m) \to +\infty$ as $n, m \to +\infty$. Two Cauchy-Gromov sequences $\{x_n\}$, $\{y_n\}$ are equivalent, denoted $(x_n) \sim (y_n)$, if $(x_n \mid y_n) \to +\infty$ as $n \to +\infty$. This is an equivalence relation if X is hyperbolic thanks to (1.1), and the Gromov boundary of X is $\partial_{\mathrm{G}} X = \{\text{Cauchy-Gromov sequences}\} / \sim$. If X is in addition proper and geodesic, this is also the visual boundary, or geodesic boundary that we will denote $\partial_\infty X$. Though not stated by Cornulier in this form, the following is given by the proof of his theorem [11, 4.3].

[3] More precisely, at first, objects are pointed metric spaces. Nevertheless the notion does not really depend on a given base-point.

Proposition 1.5. *Let u be an admissible function. Assume X and Y are hyperbolic, that X is geodesic, and let $f : X \to Y$ be a $O(u)$-Lipschitz embedding. Then f induces a (set-theoretic) boundary map $\partial_G f : \partial_G X \to \partial_G Y$. If g is $O(u)$-close to f, then $\partial_G f = \partial_G g$. If f is $O(u)$-surjective, then $\partial_G f$ is a bijection.*

This can be expressed quantitatively; we restate below certain estimates from Cornulier's proof, at the stage when the points that intervene still lie within the space. Whenever δ is a hyperbolicity constant, set a parameter

$$\mu = \begin{cases} 2^{1/\delta} & \delta > 0 \\ e & \delta = 0, \end{cases} \tag{1.2}$$

fix a base-point $o \in X$ and define a kernel $\rho_\mu : X \times X \to \mathbf{R}_{\geqslant 0}$, $\rho_\mu(x,y) := \mu^{-(x|y)_o}$. The δ-hyperbolicity inequality (1.1) translates into a quasi-ultrametric inequality for ρ_μ : $\rho_\mu(x_0,x_2) \leqslant \mu^\delta \rho(x_0,x_1) \vee \rho_\mu(x_1,x_2)$ for all $(x_0,x_1,x_2) \in X^3$. This ρ_μ can be made subadditive by the chain construction:

$$\check{\rho}_\mu(x,x') := \inf \left\{ \sum_{i=1}^n \mu^{-(x_{i-1}|x_i)_o} : n \in \mathbf{Z}_{\geqslant 1}, x = x_0, \dots x_n = x' \right\}. \tag{1.3}$$

Lemma 1.6 (Frink 1937, [18, Lemma 2][4])**.** *Let $\mathcal{X}$ be a set and $\varrho : \mathcal{X} \times \mathcal{X} \to \mathbf{R}_{\geqslant 0}$ be a $\mathbf{R}$-valued kernel on $\mathcal{X}$. Assume there is $K \in \mathbf{R}_{\geqslant 1}$ such that for all $(x_0,x_1,x_2) \in \mathcal{X}^3$, $\rho(x_0,x_2) \leqslant K\rho(x_0,x_1) \vee \rho(x_1,x_2)$. Let $\check{\varrho}$ be associated to ϱ by the chain construction (1.3). If $K \leqslant 2$, then $\check{\varrho} \leqslant \varrho \leqslant 4\check{\varrho}$.*

This allows the construction of a true distance $d_\mu = \check{\rho}_\mu$ from ρ_μ on the visual boundary. Subadditivity of the resulting kernel for points within the space plays a key role in the following result.

Theorem 1.7 (Cornulier)**.** *Let v be an admissible function. Let $(\underline{\lambda}, \overline{\lambda})$ be large-scale expansion and Lispschitz constants. Let $f : (X,o) \to (Y,o)$ be a large-scale $(\underline{\lambda}, \overline{\lambda}, v)$-sublinearly biLipschitz embedding. Assume there exists $\delta \in \mathbf{R}_{\geqslant 0}$ such that X and Y are δ-hyperbolic and that X is geodesic. For all $\alpha \in (0, \underline{\lambda})$ there exists a constant $M = M(\alpha, \delta) \in \mathbf{R}_{>0}$ and $R = R(\alpha, \lambda, v, \delta, |f(o)|) \in \mathbf{R}_{>0}$ such that for all $x, x' \in X$,*

$$(x \mid x')_o \geqslant R \implies (f(x) \mid f(x'))_o \geqslant \alpha (x \mid x')_o - M(\alpha, \delta). \tag{1.4}$$

Especially, if X and Y are proper geodesic, then $\partial_G f = \partial_\infty f$ is α-Hölder continuous for the metrics d_μ on the boundaries, where μ is set as in (1.2).

Remark 1.8. There is a dependence on μ in Cornulier's version which disappears in (1.4) because μ depends on δ according to convention (1.2).

A particular instance of theorem 1.7 occurs when the source space is $\mathbf{R}_{\geqslant 0}$ or $\mathbf{Z}_{\geqslant 0}$. For the latter, constants R and M can be explicitly extracted from the beginning of Cornulier's proof:

$$\forall s,t \in \mathbf{Z}_{\geqslant t_\alpha}, (\widetilde{\gamma}(s) \mid \widetilde{\gamma}(t))_o \geqslant \alpha \inf\{s,t\} - \log_\mu \left(\frac{2}{1 - \mu^{-(\alpha + \underline{\lambda})/2}} \right), \tag{1.5}$$

where $\widetilde{\gamma}$ replaces f of Lemma 1.7, and

$$t_\alpha = \sup \{ s \in \mathbf{Z}_{\geqslant 0} : |\widetilde{\gamma}(0)| + v(s) \geqslant 4(\underline{\lambda} - \alpha) s \} \tag{1.6}$$

[4] see also Bourbaki [2, IX.6, Proposition 2].

replaces R of 1.7. This form will be of special interest in subsection 3.B, especially the dependence of t_α on $|\widetilde{\gamma}(0)|$ is important for us.

1.C. **Metric invariants of 4 points at infinity.** Let (Y,o) be a pointed, proper geodesic hyperbolic space, and let $\partial^4_\infty Y$ denote the space of distinct 4-tuples on $\partial_\infty Y$. For $(\eta_1,\eta_2,\eta_3,\eta_4) \in \partial^4_\infty Y$, define

$$\overline{\boxtimes}\{\eta_1,\eta_2,\eta_3,\eta_4\} := \sup\{(\eta_i \mid \eta_j)_o : i \neq j\}, \text{ and}$$
$$\underline{\boxtimes}\{\eta_1,\eta_2,\eta_3,\eta_4\} := \inf\{(\eta_i \mid \eta_j)_o : i \neq j\}.$$

More generally, let (Ξ,ϱ) be a metric space (to be thought of as a geodesic boundary with a visual distance) and let $(\xi_1,\dots\xi_4)$ be distinct points in Ξ. Define their metric cross-ratio as

$$[\xi_1,\xi_2,\xi_3,\xi_4]^\varrho = [\xi_i]^\varrho := \frac{\varrho(\xi_1,\xi_3)\varrho(\xi_2,\xi_4)}{\varrho(\xi_1,\xi_4)\varrho(\xi_2,\xi_3)}.$$

The superscript ϱ might be omitted if sufficiently clear. Observe that if ϱ has been obtained by the chain construction (1.3) from a quasi-distance $\widehat{\varrho}$ on Ξ such that

$$\exists K \in [1,2),\ \forall(\xi_1,\xi_2,\xi_3) \in \Xi^3, \varrho(\xi_1,\xi_3) \leqslant K \sup\{\varrho(\xi_1,\xi_2),\varrho(\xi_2,\xi_3)\},$$

then by Frink's theorem $\varrho \leqslant \widehat{\varrho} \leqslant 4\varrho$, and

$$\forall \nu \in \mathbf{R}_{>1}, \left|\log_\nu[\xi_i] - \log_\nu \frac{\varrho(\xi_1,\xi_3)\varrho(\xi_2,\xi_4)}{\varrho(\xi_1,\xi_4)\varrho(\xi_2,\xi_3)}\right| \leqslant \log_\nu 16. \tag{1.7}$$

Especially, if $(\Xi,\varrho) = (\partial_\infty X, \check{\rho}_\nu)$ for a δ-hyperbolic, proper geodesic, pointed space (X,o) and a parameter $\nu \in (1,\mu(\delta)]$, then by (1.7), $\log_\nu[\xi_i]^{d_\nu}$ depends on ν only up to an additive error: precisely for all $\nu,\nu' \in (1,\mu]$,

$$\begin{aligned}\left|\log_\nu[\xi_i]^{d_\nu} - \log_{\nu'}[\xi_i]^{d_{\nu'}}\right| &\leqslant \left|\log_\nu[\xi_i]^{d_\nu} - (\xi_1,\xi_4)_o - (\xi_2,\xi_3)_o + (\xi_1,\xi_3)_o + (\xi_2,\xi_4)_o\right| \\ &\quad + \left|\log_{\nu'}[\xi_i]^{d_{\nu'}} - (\xi_1,\xi_4)_o - (\xi_2,\xi_3)_o + (\xi_1,\xi_3)_o + (\xi_2,\xi_4)_o\right| \\ &\leqslant \log_\nu 16 + \log_{\nu'} 16.\end{aligned}$$

In the sequel we refer to $\log_\mu[\xi_i]^{d_\mu}$ as $\log[\xi_i]$, where μ follows convention (1.2). If nonnegative, this logarithm has a geometric interpretation:

Proposition 1.9. *Let (X,o) be a proper geodesic, δ-hyperbolic space. There exists a constant $C = C(\delta)$ in $\mathbf{R}_{\geqslant 0}$ such that for all $(\xi_1,\dots\xi_4) \in \partial^4 X$,*

$$d_X(\chi_{14},\chi_{23}) - C \leqslant \log^+[\xi_i] \leqslant d_X(\chi_{14},\chi_{23}) + C.$$

where χ_{ij} are geodesic lines between ξ_i and ξ_j (whose existence is provided by the visibility property of X, see Ghys and de la Harpe [19, 7.6]).

Proposition 1.9 seems well-known, yet we could not locate a proof in the literature, so we include one in subsection 2.D. It is better understood as a statement about cross-differences, see Buyalo and Schroeder [5, 4.1].

2. Preliminaries from hyperbolic metric geometry

2.A. **A lemma on right-angled quadrilaterals.** Let $\delta \in \mathbf{R}_{\geqslant 0}$ be a constant, and let X be a geodesic δ-hyperbolic metric space. We shall work under the following convention. In the course of proofs or statements about X, one often needs to construct objects (e.g. a geodesic segment between two points). The rules of δ-hyperbolic geometry only allow to locate such objects in X up to a few multiples of δ. For us, as soon as an object in X has been constructed, it remains fixed until the end of the statement or proof so that forthcoming objects can be attached to it. This means that, for instance, if a geodesic segment between two points has been previously defined, then the midpoint of these points will be understood as the midpoint of this geodesic segment. Especially, if[5] $\gamma \subset X$ is a geodesically convex subspace and $b \in X$ is a point, $p_\gamma(b)$ is an orthogonal projection (closest point) of b on γ. This is well defined up to 16δ, and p_γ has a contracting behavior on distances expressed by the following lemma.

Lemma 2.1 (See Shchur, [27, Lemma 1][6])**.** *Let γ be a geodesic, b a point in X. Then for all $c \in \gamma$, $|c - p_\gamma(b)| \leqslant |b-c| - |b - p_\gamma(b)| + 16\delta$. In particular, for all $b, b' \in X$,*

$$|p_\gamma(b) - p_\gamma(b')| \leqslant |b - b'| + 16\delta. \tag{2.1}$$

Definition 2.2. Let $\alpha \in \mathbf{R}_{\geqslant 0}$. Say that a metric space P is α-connected if for any $\alpha' \in \mathbf{R}_{>\alpha}$, the equivalence relation generated by $[d(x,y) \leqslant \alpha']$ over $x, y \in P$ has a unique class.

Lemma 2.3. *Let $\alpha \in \mathbf{R}_{>0}$ and let $S \subset X$ be a α-connected subspace (for instance a quasigeodesic). Let γ be a geodesic of X. Then any $p_\gamma(S)$ is $(\alpha + 16\delta)$-connected. In particular if S is a geodesic then $p_\gamma(S)$ is 16δ-connected.*

Proof. Let $S' = p_\gamma(S)$ and let $\alpha' \in \mathbf{R}_{>0}$ be such that $\alpha' > \alpha + 16\delta$. If there is $s'_1 = p_\gamma(s_1)$ such that $d(s'_1, S' \setminus \{s'_1\}) \geqslant \alpha'$, then for all $s'_2 = p_\gamma(s_2)$, $|s'_1 - s'_2| > \alpha'$ implies with (2.1), that $s_1 - s_2 > \alpha'$. Thus S is not α-connected. □

Definition 2.4. Let $\eta \in \mathbf{R}_{\geqslant 0}$ be a constant and let X be a geodesic space. Say that an ordered list $x_1, \dots x_r$ of points in X with $r \geqslant 3$ is η-almost lined up if there exists a geodesic segment σ such that for all i, x_i lies in the η-neighborhood $\mathcal{N}_\eta(\sigma)$ of $\operatorname{im}(\sigma)$ and the $p_\sigma(x_i)$ are lined up in this order on σ.

Lemma 2.5 (Gromov product of almost lined up points)**.** *Let $\eta \in \mathbf{R}_{\geqslant 0}$ and let x_1, x_2, x_3 be three points in a geodesic metric space X which are η-almost lined up. Then*

$$|(x_2 \mid x_3)_{x_1} - |x_1 - x_2|| \leqslant 5\eta. \tag{2.2}$$

Proof. Let σ be a geodesic segment achieving the almost-lined upness assumption. For $i \in \{1,2,3\}$, let $y_i = p_\sigma(x_i)$. By hypothesis $|x_i - y_i| \leqslant \eta$, so by the triangle inequality $||y_i - y_j| - |x_i - x_j|| \leqslant 2\eta$; then by definition of the Gromov product $\left|(x_2 \mid x_3)_{x_1} - (y_2 \mid y_3)_{y_1}\right| \leqslant 3\eta$. Finally, y_1, y_2 and y_3 are lined up, hence $(y_2 \mid y_3)_{y_1} = |y_1 - y_2|$. The conclusion follows from the triangle inequality in $\mathbf{R}$. □

Lemma 2.6 (Right-angled triangles degenerate)**.** *Let σ be a geodesic of a geodesic hyperbolic space X, $b \in X$ and $a = p_\sigma(b)$ on σ. Let c be a point of σ. Then there exists $t \in [bc]$ such that*

1. *$|a - t| \leqslant 28\delta$*
2. *$d(t, \sigma) \leqslant 4\delta$ and $d(t, [ba]) \leqslant 4\delta$.*
3. *for any u in the subsegment $[tc]$ of $[bc]$, $d(u, \sigma) \leqslant 4\delta$.*

[5]We will abusively write γ when referring to $\operatorname{im}(\gamma)$ when γ is a (quasi)geodesic.

[6]There is a 4δ additive error term instead of our 16δ in Shchur's version, because Shchur defines a δ-hyperbolic space via Rips inequality there.

In particular, if $|b-a|$, $|c-a|$ are large enough, then b, a, c are 28δ-almost lined up in this order.

Proof. Let $\triangle$ be the geodesic triangle abc with sides $[ba]$, $[bc]$ and the subsegment $[ac]$ of σ. Set $\ell = |b-c|$ and assume $\alpha : [0,\ell] \to X$ parametrizes $[bc]$ so that $\alpha(0) = c$, $\alpha(\ell) = b$. If $\sup\{d(\alpha(s),\sigma) : s \in [0,\ell]\} \leqslant 4\delta$, set $t = b$; then (3) and (2) are automatically true, while $|a-t| = d(t,\sigma) \leqslant 4\delta \leqslant 28\delta$ so that also (1) is true. Otherwise, define

$$t = \alpha(s),\ s = \inf\left\{u \in [0,\ell], d(\alpha(u),\sigma) > 4\delta\right\}.$$

As $\triangle$ is 4δ-slim, $d(t,[ba]) \leqslant 4\delta$ while $d(t,\sigma) \leqslant 4\delta$ also. Let t_b, resp. t_c be an orthogonal projection of t on σ, resp. on $[ba]$. By the triangle inequality, $|t_c - t_b| \leqslant 4\delta + 4\delta = 8\delta$. Then $|t_b - b| \leqslant |t_c - b| + 8\delta \leqslant |b-a| + 8\delta$. By the contraction Lemma 2.1, $|t_b - a| \leqslant 8\delta + 16\delta = 24\delta$. By the triangle inequality, $|t-a| \leqslant 24\delta + 4\delta = 28\delta$. □

Lemma 2.7 (Quadrilaterals with two consecutive right-angles degenerate)**.** *Let a_0, a_1, b_0, b_1 be four points in X. For $i \in \{0,1\}$, let γ_i be a geodesic segment between a_i and b_i. Assume that $138\delta \leqslant |a_0 - a_1|$, and that one of the following holds:*

(1) *Either, $a_i = p_\sigma(b_i)$ for all $i \in \{0,1\}$, or*
(2) *$a_i = p_{\gamma_i} a_{1-i}$ for all $i \in \{0,1\}$.*

Then for all $i \in \{0,1\}$, $d(a_i,[b_0b_1]) \leqslant 56\delta$.

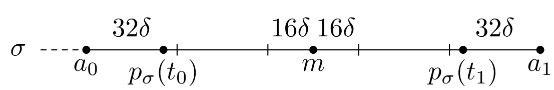
Proof. Let σ be a geodesic segment between a_0 and a_1, and let m be the midpoint of σ. By Lemma 2.6, there exists t_0 and t_1 on $[b_0m]$ and $[b_1m]$ respectively such that

$$\forall i \in \{0,1\},\ |a_i - t_i| \leqslant 28\delta \tag{2.3}$$

Moreover, by (2) of Lemma 2.6 and the triangle inequality,

$$|p_\sigma(t_i) - a_i| \leqslant |p_\sigma(t_i) - t_i| + |t_i - a_i| \leqslant 4\delta + 28\delta = 32\delta. \tag{2.4}$$

Thus a_i, $p_\sigma(t_i)$, m and a_{1-i} are lined up on σ as below:



Next, we proceed to prove that t_i is far from $[mb_{1-i}]$. Note that since the triangles ma_ib_i are slim, one need only show that t_i is far from $[a_{1-i}b_{1-i}]$ and $[ma_{1-i}]$.

- In case (1), for all $a_i' \in \gamma_i$, since $p_\sigma(a_i') = a_i$ and by (2.4) and Lemma 2.1,
$$\begin{aligned}|t_i - a_{1-i}'| &\geqslant |p_\sigma(t_i) - a_{1-i}| - 16\delta \geqslant |a_i - a_{1-i}| - |p_\sigma(t_i) - a_i| - 16\delta \\ &\geqslant 138\delta - 48\delta = 90\delta,\end{aligned}$$
hence $d(t_i,\gamma_{1-i}) \geqslant 90\delta$.
- In case (2), as $a_{1-i} = p_{\gamma_{1-i}} a_i$, $d(t_i,\gamma_{1-i}) \geqslant d(a_i,\gamma_{1-i}) - 28\delta \geqslant 110\delta$.
- In both cases, $d(t_i,[ma_{1-i}]) \geqslant 79\delta - 32\delta = 45\delta$.

Using the previous inequality together with the fact that the triangle $a_{1-i}mb_{1-i}$ is 4δ-slim,

$$d(t_i,[mb_{1-i}]) \geqslant 45\delta - 4\delta = 41\delta > 4\delta.$$

Finally, b_1mb_2 is 4δ-slim, hence $d(t_i,[b_1b_2]) \leqslant 4\delta$, and by the triangle inequality,

$$d(a_i,[b_0b_1]) \leqslant |a_i - t_i| + d(t_i,[b_0b_1]) \leqslant 28\delta + 4\delta \leqslant 56\delta.$$ □

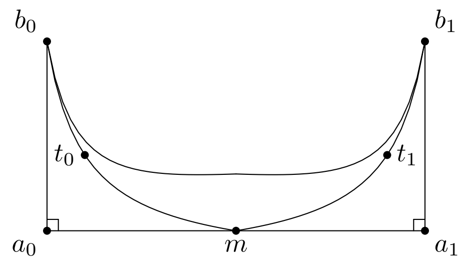


FIGURE 1. Main points occurring in the proof of Lemma 2.7.

2.B. **An estimate on geodesic projections.** Let X be as before a geodesic δ-hyperbolic metric space, and fix a base-point $o \in X$.

Lemma 2.8. *Let $\gamma, \gamma' : \mathbf{R} \to X$ be two geodesics; define $\xi_- = [\gamma]_{-\infty}$, $\xi_+ = [\gamma]_{+\infty}$, $\xi'_- = [\gamma']_{-\infty}$ and $\xi'_+ = [\gamma']_{+\infty}$ on the boundary at infinity $\partial_\infty X$ of X. Assume that the $\xi_\pm, \xi'_\pm$ are all distinct. Then*

$$\sup \{|p_\gamma(b)| : b \in \gamma'\} \leqslant \overline{\boxtimes}\{\xi_-, \xi_+, \xi'_-, \xi'_+\} + 284\delta, \tag{2.5}$$

where we recall that $\overline{\boxtimes}\{\xi_-, \xi_+, \xi'_-, \xi'_+\}$ is an abbreviation for $\sup (\xi_1 \mid \xi_2)_o$ *over distinct pairs $\{\xi_1, \xi_2\}$ in $\{\xi_\pm, \xi'_\pm\}$.*

Proof. Change if necessary the parametrizations of γ and γ' in such a way that $\gamma(0) = p_\gamma(o)$, $\gamma'(0) = p_{\gamma'}(o)$. Let $b \in \gamma'$.

- Either $|p_\gamma(b) - \gamma(0)| < 138\delta$; then by the triangle inequality, $|p_\gamma(b)| < |\gamma(0)| + 138\delta$. Let $s \in \mathbf{R}$. Since X is δ-hyperbolic,

$$(\gamma(s) \mid \gamma(-s))_o \geqslant \min \{(\gamma(-s) \mid \gamma(0))_o, (\gamma(0) \mid \gamma(s))_o\} - \delta. \tag{2.6}$$

  By Lemma 2.6, when s is large enough o, $\gamma(0)$ and $\gamma(s)$ (resp. o, $\gamma(0)$ and $\gamma(s)$) are 28δ-almost lined up in this order, so by Lemma 2.5, (2.6) becomes

$$(\gamma(s) \mid \gamma(-s))_o \geqslant |\gamma(0)| - 5 \cdot 28\delta - \delta = |\gamma(0)| - 141\delta.$$

  Finally, $|p_\gamma(b)| < |\gamma(0)| + 138\delta \leqslant (\gamma(s) \mid \gamma(-s))_o + 138\delta + 141\delta$. Letting $s \to +\infty$,

$$\begin{aligned} |p_\gamma(b)| &\leqslant \liminf_{s \to +\infty} (\gamma(s) \mid \gamma(-s))_o \leqslant (\xi_- \mid \xi_+)_o + 279\delta \\ &\leqslant (\xi_- \mid \xi_+)_o + 284\delta. \end{aligned}$$

- Or $|p_\gamma(b) - \gamma(0)| \geqslant 138\delta$ in which case Lemma 2.7 applies so that o, $\gamma(0)$ and $p_\gamma(b), b$ are 56δ-almost lined up in this order. Let $s, s' \in \mathbf{R}$ be such that $\inf\{|s|, |s'|\} \geqslant \sup\{|p_\gamma(b) - \gamma(0)|, |b - \gamma'(0)|\}$. Then

$$\begin{aligned} (\gamma(s) \mid \gamma'(s'))_o \geqslant \min \{&(\gamma(s) \mid \gamma(0))_o, (\gamma(0), p_\gamma(b))_o, \\ &(p_\gamma(b) \mid b)_o, (b \mid \gamma'(0))_o, (\gamma'(0) \mid \gamma'(s'))_o\} - 4\delta. \end{aligned}$$

  Applying repeatedly Lemma 2.5,

$$\begin{aligned} (\gamma(s) \mid \gamma'(s'))_o \geqslant \min \{&|\gamma(0)| - 140\delta, |\gamma(0)| - 140\delta, \\ &|p_\gamma(b)| - 5 \cdot 56\delta, |\gamma'(0)| - 140\delta, |\gamma'(0)| - 140\delta\} - 4\delta. \end{aligned}$$

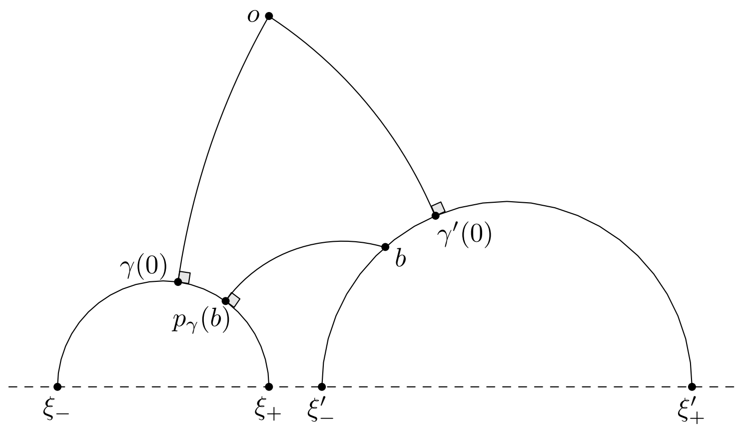


FIGURE 2. Configuration of Lemma 2.8 in the half-plane model of $\mathbb{H}^2$.

Now letting $s, s' \to \pm\infty$,

$$\begin{aligned}|p_\gamma(b)| &\leqslant \overline{\boxtimes}\{\xi_-, \xi'_-, \xi_+, \xi'_+\} + 5 \cdot 56\delta + 4\delta \\ &= \overline{\boxtimes}\{\xi_-, \xi'_-, \xi_+, \xi'_+\} + 284\delta. \end{aligned}$$ □

2.C. **Quantitative Morse stability.** We prove here a version of the Morse lemma with an emphasis on the linear dependence of the tracking distance on the quasi-isometry additive error term.

Lemma 2.9 (Morse stability for quasigeodesics)**.** *Let $c, \delta \in \mathbf{R}_{\geqslant 0}$, $(\underline{\lambda}, \overline{\lambda}) \in \mathbf{R}^2_{>0}$ be constants. Let X be a geodesic, δ-hyperbolic metric space. Let $J = [a, b]$ be a closed bounded interval of $\mathbf{R}$ and let $\widetilde{\gamma} : J \to X$ be $(\underline{\lambda}, \overline{\lambda}, c)$ quasigeodesic, i.e.*

$$\forall (s,t) \in J^2, \underline{\lambda}|s-t| - c \leqslant |\widetilde{\gamma}(s) - \widetilde{\gamma}(t)| \leqslant \overline{\lambda}|s-t| + c.$$

Recall that $\lambda = \sup\{\overline{\lambda}, 1/\underline{\lambda}\}$, and assume that[7] *$c \geqslant 6\lambda^2\delta$. There exist functions $h, \widetilde{h} : \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ such that if $\gamma : [0, |\widetilde{\gamma}(b) - \widetilde{\gamma}(a)|] \to X$ is any geodesic segment with same endpoints as $\widetilde{\gamma}$, then*

$$\forall t \in J, d(\widetilde{\gamma}(t), \operatorname{im}(\gamma)) \leqslant h(\lambda)(\delta + c) \tag{2.7}$$

$$\forall s \in [0, |\widetilde{\gamma}(b) - \widetilde{\gamma}(a)|], d(\gamma(s), \operatorname{im}(\widetilde{\gamma})) \leqslant \widetilde{h}(\lambda)(\delta + c). \tag{2.8}$$

Precisely, h and $\widetilde{h}$ can be taken as $h(\lambda) = 12(1 + 8\lambda^2)$ and $\widetilde{h}(\lambda) = 16(5 + 6\lambda^2)$.

Remark 2.10. Our expression for $\widetilde{h}(\lambda)$ is certainly not optimal: Shchur [27, Theorem 2] claims that $\widetilde{h}(\lambda) = O(\log \lambda)$. For us in the following, only the linear dependence over the sum of additive errors $\delta + c$ in (2.7) and (2.8) matters.

Proof. A sketch of proof for the part of lemma expressed by (2.7) can be found in Thurston's exposition of the Mostow rigidity theorem, [28, 5.9.2] with non-explicit right-hand side bound; see also an early (and more explicit) proof by Efremovich and Tihomirova [16, p. 1142–1143], also taking place in $\mathbb{H}^n_{\mathbf{R}}$. When projecting onto

[7] This assumption could be dropped; we make it in order to simplify h and $\widetilde{h}$, and because c is to be replaced by an unbounded function v in the next section.

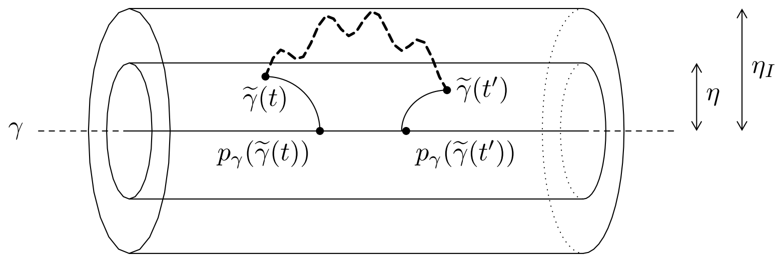


FIGURE 3. Proof of the Morse stability lemma 2.9.

a geodesic line in hyperbolic space, the lengths of curves situated at a distance η are contracted with a factor depending exponentially[8] on η, so that the length of portions of quasigeodesics leaving a tube of thickness η around a geodesic can be bounded. This can be carried into a general argument in δ-hyperbolic space, replacing length by a rough analogue; for this we build on Shchur's work [27]. For $\alpha \in \mathbf{R}_{>0}$, $I \subset \mathbf{R}$ a bounded interval and $\sigma : I \to X$ a curve such that $\sigma(I)$ is $\alpha/2$-connected, define the length of σ at scale α as

$$\ell_\alpha(\sigma) = \sup_{(t_i) \in T_\alpha(\sigma)} \sum_i |\sigma(t_i+1) - \sigma(t_i)|,$$

where $(t_i) \in T_\alpha(\sigma)$ if there is $r \in \mathbf{Z}_{\geqslant 0}$ such that $\inf I = t_0 < \cdots < \sup I = t_r$ and if $\{\sigma(t_i)\}$ is a α-separated net in $\operatorname{im}(\sigma)$. If σ is a $(\underline{\lambda}, \overline{\lambda}, c)$-quasi-geodesic segment (e.g. a portion of $\widetilde{\gamma}$) and α is such that $\alpha \geqslant 2c$, then

$$\ell_\alpha(\sigma) \leqslant 2\overline{\lambda}|I|, \tag{2.9}$$

see Shchur [27, Lemma 7]. Now let η be a positive real number (to be fixed later). Define $\mathcal{N}_\eta\gamma$ as the η-neighborhood of $\operatorname{im}(\gamma)$ in X, and

$$U_\eta = \{t \in J, \widetilde{\gamma}(t) \notin \mathcal{N}_\eta\gamma\}.$$

Let $I \in \pi_0(U_\eta)$, $t = \inf I$ and $t' = \sup I$. t and t' are both finite, since J is bounded and $\widetilde{\gamma}$ and γ have the same endpoints. Then $\widetilde{\gamma}_{|[t,t']}$ is outside $\mathcal{N}_\eta(\gamma)$; by Shchur's exponential contraction estimate[9] [27, Lemma 10], there exists a constant $S \in \mathbf{R}_{>0}$ such that, as soon as $\eta \geqslant 2c + 12\delta$,

$$\begin{aligned} |p_\gamma\widetilde{\gamma}(t) - p_\gamma\widetilde{\gamma}(t')| &\leqslant \sup\left\{\frac{6\delta}{c}e^{-S\eta}\ell_{2c}\widetilde{\gamma}_{|[t,t']}, 24\delta\right\} \\ &\leqslant 24\delta + \frac{6\delta}{c}e^{-S\eta}\ell_{2c}\widetilde{\gamma}_{|[t,t']}. \end{aligned} \tag{2.10}$$

On the other hand,

$$\begin{aligned} \ell_{2c}\widetilde{\gamma}_{|[t,t']} \underset{(2.9)}{\leqslant} 2\overline{\lambda}|t'-t| &\leqslant 2(\overline{\lambda}/\underline{\lambda})\left[|\widetilde{\gamma}(t') - \widetilde{\gamma}(t)| + c\right] \\ &\leqslant 2\lambda^2\left[2\eta + |p_\gamma\widetilde{\gamma}(t) - p_\gamma\widetilde{\gamma}(t')| + c\right], \end{aligned} \tag{2.11}$$

[8] It is useful to write the hyperbolic metric in cylindrical coordinates around γ to appreciate that the contraction factor is a hyperbolic cosine of η.

[9] Shchur's lemma is actually stated in a slightly different form, namely our $|p_\gamma\widetilde{\gamma}(t) - p_\gamma\widetilde{\gamma}(t')|$ is replaced by $\operatorname{diam} p_\gamma\widetilde{\gamma}(I)$, and follows a different convention on the δ hyperbolicity constant.

where we used the triangle inequality together with the fact that $\widetilde{\gamma}(t), \widetilde{\gamma}(t') \in \partial\mathcal{N}_\eta\gamma$ for the last inequality. Combining (2.10) and (2.11),

$$\begin{aligned}\frac{1}{2\lambda^2}\ell_{2c}\widetilde{\gamma}_{|[t,t']} &\leqslant 2\eta + |p_\gamma\widetilde{\gamma}(t) - p_\gamma\widetilde{\gamma}(t')| + c\\ &\leqslant 2\eta + 24\delta + \frac{6\delta}{c}e^{-S\eta}\ell_{2c}\widetilde{\gamma}_{|[t,t']} + c,\end{aligned}$$

hence

$$\left(\frac{1}{2\lambda^2} - \frac{6\delta}{c}e^{-S\eta}\right)\ell_{2c}\widetilde{\gamma}_{|[t,t']} \leqslant 2\eta + 24\delta + c. \tag{2.12}$$

Define $\eta_I = \sup_{u\in I} d(\widetilde{\gamma}(u), \gamma)$. Then, as $c > 3\delta\lambda^2$ by hypothesis,

$$\eta_I \leqslant \eta + \frac{1}{2}\ell_{2c}\widetilde{\gamma}_{|[t,t']} \vee 2c \underset{(2.12)}{\leqslant} \eta + 2c + \frac{2\eta + 24\delta + c}{1/\lambda^2 - (3\delta/c)\cdot e^{-S\eta}}. \tag{2.13}$$

It remains to set η in order to explicit the bound on η_I given by the last inequality. Actually, as $c \geqslant 6\delta\lambda^2$, if $\eta = 2c + 12\delta$ (remember that $\widetilde{\gamma}_{|I}$ must be at least this far for the exponential contraction to operate),

$$\begin{aligned}\eta_I &\leqslant \eta + 2c + \frac{2\eta + 24\delta + c}{1/(2\lambda^2)}\\ &\leqslant 12\delta + 4c + \lambda^2\left(4\eta + 48\delta + c\right)\\ &= 12\delta + 4c + \lambda^2(96\delta + 5c) \leqslant 12(1 + 8\lambda^2)(\delta + c).\end{aligned}$$

Finally,

$$\begin{aligned}\sup\{d(\widetilde{\gamma}(t), \gamma) : t \in J\} = \eta \vee \sup_{I\in\pi_0 U_\eta} \eta_I &\leqslant 12(\delta + c) \vee 12(1 + 8\lambda^2)(\delta + c)\\ &= 12(1 + 8\lambda^2)(\delta + c).\end{aligned}$$

This is (2.7). Now, let $s \in [0, |\widetilde{\gamma}(b) - \widetilde{\gamma}(a)|]$. Because $\widetilde{\gamma}$ is c-connected, by Lemma 2.3 $p_\gamma\widetilde{\gamma}$ is $c+16\delta$-connected, so there is $s' \in [0, |\widetilde{\gamma}(b) - \widetilde{\gamma}(a)|]$ such that $|s' - s| \leqslant c + 16\delta$ and $s' = p_\gamma(\widetilde{\gamma}(t))$ for a $\hat{t} \in J$. The triangle inequality in X yields

$$\begin{aligned}d(\gamma(s), \mathrm{im}(\widetilde{\gamma})) \leqslant |\gamma(s) - \widetilde{\gamma}(\hat{t})| &\leqslant |s - s'| + |\gamma(s') - \widetilde{\gamma}(\hat{t})|\\ &\underset{(2.7)}{\leqslant} 12(1 + 8\lambda^2)(\delta + c) + 16\delta + c\\ &\leqslant 16(5 + \lambda^2)(\delta + c).\end{aligned}$$

This is (2.8). □

Remark 2.11. V. Shchur [27, Theorem 1] claims a stronger result. However the proof in [27] has a gap, noticed by S. Gouëzel and recently fixed by Gouëzel and Shchur.

2.D. **Proof for Proposition 1.9.** Let $\xi_1, \ldots \xi_4$ be as in the statement of Proposition 1.9 and assume that the geodesic lines χ_{14} and χ_{23} are parametrized in such a way that a common perpendicular geodesic segment σ falls on $\chi_{14}(0)$ and $\chi_{23}(0)$, accordingly to figure 4. Let H be the metric subspace of X defined as $\chi_{14} \cup \chi_{23} \cup \sigma$ and denote by $|\cdot|_\mathsf{H}$ the path distance in H. By Lemma 2.7 (1), if $d(\chi_{14}, \chi_{23}) \geqslant 138\delta$ then for all $t \in \mathbf{R}$, whenever $(\chi, \chi') \in \{\chi_{14}, \chi_{23}\}^2$,

$$||\chi(t) - \chi'(\epsilon t)| - |\chi(t) - \chi'(\epsilon t)|_\mathsf{H}| \leqslant 4 \cdot 56\delta = 212\delta. \tag{2.14}$$

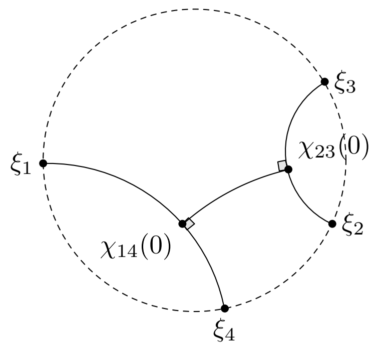


FIGURE 4. Geometric interpretation of the nonnegative part of the logarithm of cross-ratio: up to an additive error, this is the distance $|\chi_{14}(0) - \chi_{23}(0)|$.

For all $t \in \mathbf{R}$ (compare Buyalo and Schroeder [5, p. 37]),

$$2\left\{\begin{array}{c}(\chi_{14}(-t) \mid \chi_{14}(t))_o \\ +(\chi_{23}(t) \mid \chi_{23}(-t))_o \\ -(\chi_{14}(-t) \mid \chi_{23}(t))_o \\ -(\chi_{14}(t) \mid \chi_{23}(-t))_o\end{array}\right\} = \left\{\begin{array}{c}|\chi_{14}(-t)| + |\chi_{14}(t)| - 2t \\ +|\chi_{23}(-t)| + |\chi_{23}(t)| - 2t \\ -|\chi_{14}(-t)| + |\chi_{23}(t)| + |\chi_{14}(-t) - \chi_{23}(t)| \\ -|\chi_{14}(t)| - |\chi_{23}(-t)| + |\chi_{14}(t) - \chi_{23}(-t)|\end{array}\right\}$$

$$= -4t + |\chi_{14}(-t) - \chi_{23}(t)| + |\chi_{14}(t) - \chi_{23}(-t)|. \tag{2.15}$$

By (2.14), there is Δ with $|\Delta| \leqslant 2 \cdot 212\delta = 424\delta$ such that

$$\begin{aligned} &-4t + |\chi_{14}(-t) - \chi_{23}(t)| + |\chi_{14}(t) - \chi_{23}(-t)| \\ &= -4t + |\chi_{14}(-t) - \chi_{23}(t)|_{\mathsf{H}} + |\chi_{14}(t) - \chi_{23}(-t)|_{\mathsf{H}} + \Delta \\ &= 2d(\chi_{14}, \chi_{23}) + \Delta. \end{aligned} \tag{2.16}$$

On the other hand, by (1.7),

$$\left| \log_\mu[\xi_i] - \lim_{t \to +\infty} \left\{ \begin{array}{c}(\chi_{14}(-t) \mid \chi_{14}(t))_o + (\chi_{23}(t) \mid \chi_{23}(-t))_o \\ -(\chi_{14}(-t) \mid \chi_{23}(t))_o - (\chi_{14}(t) \mid \chi_{23}(-t))_o\end{array}\right\} \right| \leqslant 8\delta + \log_\mu 16.$$

If $d(\chi_{14}, \chi_{23})$ is large enough, letting $t \to +\infty$ in (2.15) combined with the estimate (2.16), we reach the desired inequality of Proposition 1.9. This is valid for small values as well since $\log^+$ then takes small values.

Remark 2.12. The right-hand side inequality of Proposition 1.9 can be deduced from the elementary case of a metric tree via tree approximation [19, Theorem 2.12]. See Bourdon's remark [4, 2.3].

3. Sublinear tracking

Sublinearly biLipschitz embeddings of the real half-line, resp. of the real line admit trackings by geodesic rays, resp. lines; we prove this in 3.B, resp. 3.C. In the spirit of (2.7) and (2.8), the bound on the tracking distance can be expressed as a constant (denoted $H, \widetilde{H}$...) times the additive error function v, however at the cost of being valid only farther than a given tracking radius. The tracking constants and the tracking radii depend on v, more precisely through its large-scale features $v \uparrow \tau$, $r_\varepsilon(v)$ and $\sup\{r : v(r) \leqslant \mathrm{cst}(\lambda, \delta, \ldots)\}$ described in 1.A. While the use of tracking radii allow tracking estimates to take a particularly simple form when applied in 3.D, their dependence upon v must not be kept entirely implicit, especially it must be taken into account for later use in section 4 when v becomes a parameter, a task undertaken in 3.E.

3.A. **Preliminaries.** Unless otherwise stated, geodesic rays into a pointed metric space are assumed to have their origin at the base-point. This convention will not apply to the rougher $O(v)$-rays that we define hereafter.

Definition 3.1. Let u be an admissible function and X a metric space. A $O(u)$-geodesic, resp. a $O(u)$-ray in X is a $O(u)$-sublinearly biLipschitz embedding $\mathbf{R} \to X$, resp. $\mathbf{R}_{\geqslant 0} \to X$.

When $u = 1$, this is the classical notion of a quasigeodesic, resp. of a quasi-geodesic ray. By definition, $O(u)$-geodesics, resp. $O(u)$-rays, are sent to $O(u)$-geodesics resp. $O(u)$-rays when one applies a $O(u)$-sublinearly biLipschitz embedding to the space. $O(u)$-geodesics behave like quasi-geodesic inside every ball, with an additive error parameter controlled by the radius; however the containing ball sits in the target space, so that the dependence of the additive error on radius only becomes apparent on the large scale. We turn this observation into a lemma, which may be considered as an alternative definition for large-scale Lipschitz embeddings, easier to handle through certain technical steps.

Lemma 3.2. *Let u be an admissible function. Let $(\underline{\lambda}, \overline{\lambda})$ be Lipschitz constants, let $v = O(u)$ be nondecreasing, and let $f : (X, o) \to (Y, o)$ be a large-scale $(\underline{\lambda}, \overline{\lambda}, v)$-biLipschitz embedding. Then there exist $\widehat{v} = O(u)$, $t_{\bigcirc} \in \mathbf{R}_{\geqslant 0}$ and $R_{\bigcirc} \in \mathbf{R}_{\geqslant 0}$ (depending on f and v) such that for all $x, x_1, x_2 \in X$*

(I) *If $x \notin B(o, t_{\bigcirc})$ or $f(x) \notin B(o, R_{\bigcirc})$ then*

$$\frac{1}{3\lambda}|x| \leqslant |x| \wedge |f(x)| \leqslant 3\lambda|x|.$$

(II) *If $x_1, x_2 \in X \setminus B(o, t_{\bigcirc})$ or $f(x_1), f(x_2) \in Y \setminus B(o, R_{\bigcirc})$, then*

$$\begin{cases} |f(x_1) - f(x_2)| \leqslant \lambda|x_1 - x_2| + \widehat{v}\left((|x_1| \vee |x_2|) \wedge (|f(x_1)| \vee |f(x_2)|)\right), \\ |f(x_1) - f(x_2)| \geqslant \frac{1}{\lambda}|x_1 - x_2| - \widehat{v}\left((|x_1| \vee |x_2|) \wedge (|f(x_1)| \vee |f(x_2)|)\right). \end{cases}$$

Moreover $t_{\bigcirc}$, $R_{\bigcirc}$ and $\widehat{v}$ may be taken as:

$$t_{\bigcirc}(|f(o)|, v) = \sup\left\{ r : v(r) \geqslant \frac{r}{3\lambda} \right\} \vee 3\lambda|f(o)| = r_{1/(3\lambda)}(v) \vee 3\lambda|f(o)|, \tag{3.1}$$

$$R_{\bigcirc}(|f(o)|, v) = 4|f(o)| \vee 2(2\lambda + 1)t_{\bigcirc}(|f(o)|, v), \text{ and} \tag{3.2}$$

$$\widehat{v} = (v \uparrow 3\lambda)v. \tag{3.3}$$

Proof. By definition of $t_{\bigcirc}(|f(o)|, v)$, for all $x \in X \setminus B(o, t_{\bigcirc})$, $|f(o)| \leqslant 1/(3\lambda)|x|$ and $v(|x|) \leqslant \frac{1}{3\lambda}|x|$, so $\frac{1}{3\lambda}|x| \leqslant |f(x)| \leqslant \left(\lambda + \frac{2}{3\lambda}\right)|x| \leqslant 3\lambda|x|$; this is the first case in (I). Now assume that $R_{\bigcirc}$ is defined as in (3.2). Note that $R_{\bigcirc} \geqslant 2r_{1/(3\lambda)}(v) \geqslant r_{1/2}(v)$ so that if $f(x) \in Y \setminus B(o, R_{\bigcirc}(|f(o)|, v))$, then

$$\begin{aligned} |x| &\geqslant \overline{\lambda}^{-1}\left(|f(x)| - |f(o)| - v(|x|)\right) \\ &\geqslant \begin{cases} \lambda^{-1}(|f(x)| - |f(o)| - v(|f(x)|) & \text{if } |x| \leqslant |f(x)|, \text{ or} \\ \lambda^{-1}(|f(x)| - |f(o)| - |x|/2) & \text{if } |x| \geqslant |f(x)|. \end{cases} \end{aligned}$$

In both cases,

$$|x| \geqslant \frac{1}{\lambda + 1/2}\left(\frac{1}{2}|f(x)| - |f(o)|\right),$$

and then $|x| \geqslant t_{\bigcirc}(|f(o)|, v)$ since by definition $R_{\bigcirc} \geqslant 2|f(o)|+(2\overline{\lambda}+1)t_{\bigcirc}$. Hence the hypotheses in (I) actually reduce to the single first one. (II) follows from (I), the fact that f is a (λ, v)-embedding, that $\widehat{v}$ is nondecreasing, and the left distributivity of $\leqslant$ over $\wedge$. □

3.B. **Rays.** Let Y be a proper geodesic hyperbolic space, and $\widetilde{\gamma} : \mathbf{R} \to Y$ a $O(u)$-geodesic ray. Inequality (1.5) says in particular that $\{\widetilde{\gamma}(t)\}_{t\in\mathbf{Z}_{\geqslant 0}}$ is a Cauchy-Gromov sequence. Since Y is proper and geodesic, its Gromov boundary is equal to $\partial_\infty Y$ and there exists a geodesic ray $\gamma : (\mathbf{R}_{\geqslant 0}, 0) \to (Y, o)$ such that $\eta := [\gamma] = \partial_\infty \widetilde{\gamma}(+\infty)$. We will prove that γ actually tracks $\widetilde{\gamma}$, in the sense that the growth of distance between them is in the $O(u)$-class. We need a preliminary lemma.

Lemma 3.3. *Let $\delta \in \mathbf{R}_{\geqslant 0}$, and let (Y, o) be a proper geodesic δ-hyperbolic space. Let $\gamma : \mathbf{R} \to Y$ be a geodesic ray into Y, and let γ' be a non-pointed geodesic ray asymptotic to γ, i.e. $[\gamma] = [\gamma'] \in \partial_\infty Y$. Then for all $s \in \mathbf{R}_{\geqslant 0}$ such that $s \geqslant |\gamma'(0)| + 16\delta$,*

$$d(\gamma'(s), \mathrm{im}(\gamma)) \leqslant 8\delta.$$

Proof. This is a classical result in hyperbolic metric geometry, use for instance the proof of $(ii) \implies (iii)$ in [19, Proposition 7.1] with appropriate changes of notation, and replace Ghys and de la Harpe's D with $\sup\{|\gamma'(0)|, 16\delta\}$. □

Lemma 3.4 (Sublinear tracking for rays)**.** *Let v be an unbounded admissible function. Let (Y, o) be a proper, geodesic, pointed metric space. Assume there exists $\delta \in \mathbf{R}_{\geqslant 0}$ such that Y is δ-hyperbolic. Let $(\underline{\lambda}, \overline{\lambda}) \in \mathbf{R}^2_{>0}$ be Lipschitz data, and let $\widetilde{\gamma} : \mathbf{R}_{\geqslant 0} \to Y$ be a $(\underline{\lambda}, \overline{\lambda}, v)$-ray. Let $\eta \in \partial_\infty Y$ be the endpoint of $\widetilde{\gamma}$, and let γ be any geodesic ray such that $[\gamma] = \eta$. Then there exist constants $H, \widetilde{H} \in \mathbf{R}_{>0}$, $t_{\backsimeq}, R_{\backsimeq}, \in \mathbf{R}_{\geqslant 0}$ such that for all positive real t and s,*

$$t \geqslant t_{\backsimeq} \implies d(\widetilde{\gamma}(t), \gamma) \leqslant Hv(t) \tag{3.4}$$

$$s \geqslant R_{\backsimeq} \implies d(\gamma(s), \mathrm{im}(\widetilde{\gamma})) \leqslant \widetilde{H}v(s), \tag{3.5}$$

where H and $\widetilde{H}$ depend on λ and v only, while $t_{\backsimeq}$ and $R_{\backsimeq}$ can be decomposed into

$$t_{\backsimeq} = t^0_{\backsimeq}(\lambda, v, \delta) + 2\lambda|\widetilde{\gamma}(0)| \tag{3.6}$$

$$R_{\backsimeq} = R^0_{\backsimeq}(\lambda, v, \delta) + |\widetilde{\gamma}(0)|. \tag{3.7}$$

Remark 3.5. In view of Lemma 2.9, it does matter for us that v be unbounded. If v is bounded, though, $\widetilde{\gamma}$ is a quasi-geodesic ray and the same result classically holds, see for instance Ghys and de la Harpe [19, 5.25], with extra additive terms in the estimates (3.4) and (3.5).

Remark 3.6. It is important to make the dependence of the tracking radius $R^0_{\backsimeq}$ upon the function v explicit, at least to some extent. However, in order not to overload the current proof, we reconstruct it separately (but along with other tracking radii) in subsection 3.E, and only keep record of the steps needed for its definition here, with enough details to ensure that it only depends on λ, v and δ.

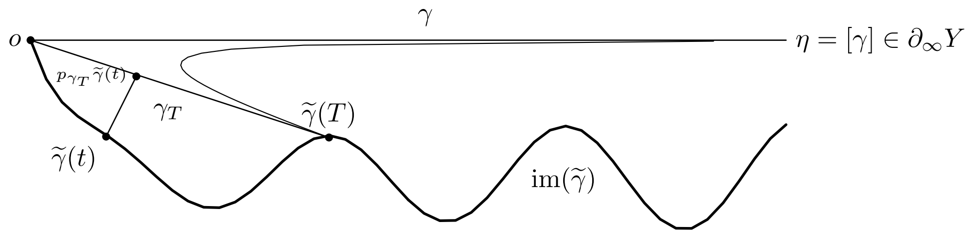


FIGURE 5. Sublinear tracking for $O(u)$-rays (Step 1: $o = \widetilde{\gamma}(0)$).

Sketch of proof for Lemma 3.4. For every $t \in \mathbf{R}_{\geqslant 0}$, set a real positive T large enough according to t so that (1.5) ensures the Gromov product $(\widetilde{\gamma}(T), \eta)_o$ is significantly greater than $|\widetilde{\gamma}(t)|$, and use the stability lemma 2.9 to prove that $\widetilde{\gamma}(t)$ is not far from the geodesic segment γ_T between o and $\widetilde{\gamma}(T)$. Here keeping an efficient inequality requires that T stay within linear control of t, which can be done consistently with the antagonist constraint of (1.5). Further, show that the projection of $\widetilde{\gamma}(t)$ on γ_T is close to γ, using the slim triangle $o\widetilde{\gamma}(T)\eta$, see figure 5. Finally, (3.5) is deduced from (3.4) with a metric connectedness argument in the same way that (2.8) was deduced from (2.7) in the proof of Lemma 2.9.

Proof of lemma 3.4. Setting $\alpha = \underline{\lambda}/2$ in the Gromov product estimate (1.5) and letting $s \to +\infty$,

$$\forall T \in \mathbf{Z}_{\geqslant t_\alpha},\ ([\gamma] \mid \widetilde{\gamma}(T))_o \geqslant \underline{\lambda}T/2 - M'(\underline{\lambda}, \delta), \tag{3.8}$$

where $M'(\underline{\lambda}, \delta) = \log_\mu \left(\frac{2}{1-\mu^{-(3\underline{\lambda})/4}}\right)$. We will first prove the lemma in the case $|\widetilde{\gamma}(0)| = 0$, i.e. $\widetilde{\gamma}(0) = o$ (this is pictured on Figure 5), and then use 3.3 to extend the result to the general case.

Step 1: $\widetilde{\gamma}(0) = o$. — Let $(t, T) \in \mathbf{R}^2_{\geqslant 0}$ be such that $t \leqslant T$. Since v is nondecreasing and unbounded, there is $T_2 \in \mathbf{R}_{>0}$ such if $T \geqslant T_2$, then $v(T) \geqslant 6\lambda^2\delta$. This is the condition required to apply Lemma 2.9. By inequality (2.7) of this lemma applied to $\gamma_T = [o\widetilde{\gamma}(T)]$ and $\widetilde{\gamma}_{|[0,T]}$,

$$\text{if } T \geqslant T_2,\ d(\widetilde{\gamma}(t), \mathrm{im}(\gamma_T)) \leqslant h(\lambda)\,(\delta + v(T))\,. \tag{3.9}$$

Similarly, by (2.8), if $T \geqslant T_2$ then

$$\forall S \in [0, |\widetilde{\gamma}(T)|],\ d(\gamma_T(S), \mathrm{im}(\widetilde{\gamma})) \leqslant \widetilde{h}(\lambda)\,(\delta + v(T))\,. \tag{3.10}$$

By (1.6) and our definition of α, $t_\alpha = r_{2\underline{\lambda}}(v)$; start assuming that $t \geqslant t_\alpha \vee T_2$. We look for T greater than t (hence, greater than t_α and T_2) such that $(\widetilde{\gamma}(T) \mid \eta)_o \geqslant 2|\widetilde{\gamma}(t)|$. Thanks to (3.8) this holds when $T = \lceil t \rceil \vee \lceil (4/\underline{\lambda})\,(|\widetilde{\gamma}(t)| + M(\underline{\lambda}, \delta)) \rceil$; we keep this dependence of T with respect to t from now on. Let $\triangle_T$ be a (geodesic, semi-ideal) triangle with vertices o, $\widetilde{\gamma}(T)$ and η (Recall that by convention, γ_T is the side of $\triangle_T$ between o and $\widetilde{\gamma}(T)$). By (3.9) and the triangle inequality,

$$|p_{\gamma_T}(\widetilde{\gamma}(t))| \leqslant h(\lambda)\,(\delta + v(T)) + |\widetilde{\gamma}(t)|.$$

Again by the triangle inequality,

$$\begin{aligned} d\left(p_{\gamma_T}\widetilde{\gamma}(t), [\widetilde{\gamma}(T)\eta)\right) &\geqslant \left|p_{[\widetilde{\gamma}(T)\eta)}o\right| - \left|p_{\gamma_T}(\widetilde{\gamma}(t))\right| \\ &\geqslant \left|p_{[\widetilde{\gamma}(T)\eta)}o\right| - h(\lambda)\left(\delta + v(T)\right) - |\widetilde{\gamma}(t)|. \end{aligned}$$

By the triangle inequality $\left|p_{[\widetilde{\gamma}(T)\eta)}o\right| \geqslant (\widetilde{\gamma}(T) \mid \eta)_o$, so that the previous inequality becomes

$$\begin{aligned} d\left(p_{\gamma_T}\widetilde{\gamma}(t), [\widetilde{\gamma}(T)\eta)\right) &\geqslant (\widetilde{\gamma}(T) \mid \eta)_o - h(\lambda)\left(\delta + v(T)\right) - |\widetilde{\gamma}(t)| \\ &\geqslant |\widetilde{\gamma}(t)| - h(\lambda)\left(\delta + v(T)\right), \end{aligned} \tag{3.11}$$

where we have replaced the Gromov product according to the definition of T. Let us now bound $v(T)$. By definition,

$$\begin{aligned} T &\leqslant 4\lambda|\widetilde{\gamma}(t)| + 4\lambda M'(\underline{\lambda}, \delta) + 1 \\ &\leqslant 4\lambda\left(\lambda t + v(t)\right) + 4\lambda M'(\underline{\lambda}, \delta) + 1, \end{aligned}$$

hence for $t \geqslant t_0 = \sup\{t_\alpha, T_2, 4\lambda M'(\underline{\lambda}, \delta) + 1\}$, $T \leqslant (1 + 8\lambda^2)t$, and

$$v(T) \leqslant (v \uparrow 1 + 8\lambda^2)v(t). \tag{3.12}$$

Substituting this in inequality (3.11), for all t such that $t \geqslant t_0$,

$$d\left(p_{\gamma_T}\widetilde{\gamma}(t), [\widetilde{\gamma}(T)\eta)\right) \geqslant |\widetilde{\gamma}(t)| - h(\lambda)(\delta + (v \uparrow 1 + 8\lambda^2)v(t)).$$

Applying Lemma 3.2 to $\widetilde{\gamma}$, define

$$t_1 := t_{\bigcirc} \vee \sup\{s : v(s) \leqslant \delta\} \vee 3\lambda r_{1/(2h(\lambda)v\uparrow 1+8\lambda^2)}(v) \vee 12\lambda\delta \vee t_0.$$

Then by definition of t_1,

$$\forall t \in \mathbf{R}_{>t_1},\ d\left(p_{\gamma_T}\widetilde{\gamma}(t), [\widetilde{\gamma}(T)\eta)\right) > 4\delta.$$

But $\triangle_T$ is 4δ-slim, so

$$\forall t \in \mathbf{R}_{>t_1},\ d\left(p_{\gamma_T}\widetilde{\gamma}(t), \gamma\right) \leqslant 4\delta. \tag{3.13}$$

By the triangle inequality,

$$\begin{aligned} \forall t \in \mathbf{R}_{>t_1},\ d\left(\widetilde{\gamma}(t), \mathrm{im}(\gamma)\right) &\leqslant |\widetilde{\gamma}(t) - p_{\gamma_T}\widetilde{\gamma}(t)| + d\left(p_{\gamma_T}\widetilde{\gamma}(t), \mathrm{im}(\gamma)\right) \\ &\underset{(3.9),\,(3.13)}{\leqslant} h(\lambda)\left(\delta + v(T)\right) + 4\delta \\ &\underset{(3.12)}{\leqslant} h(\lambda)\left(\delta + (v \uparrow 1 + 8\lambda^2)v(t)\right). \end{aligned}$$

Define $t_3 = \sup\{s : v(s) \leqslant h(\lambda)\delta\} \vee t_1$. The last inequality implies

$$\forall t \in \mathbf{R}_{\geqslant t_3},\ d(\widetilde{\gamma}(t), \mathrm{im}(\gamma)) \leqslant \left(2h(\lambda)(v \uparrow 1 + 8\lambda^2) + 1\right) v(t). \tag{3.14}$$

We have proved (3.4) in the special case $|\widetilde{\gamma}(0)| = 0$; set $H_0(\lambda, v) = 2h(\lambda)(v \uparrow 1 + 8\lambda^2) + 1$.

Step 2: $\widetilde{\gamma}(0)$ arbitrary. — Let γ' be a non-pointed geodesic ray $[\widetilde{\gamma}(0)\eta)$. Apply (3.14) to $\widetilde{\gamma}$ and γ'. This gives the existence, for all $t \in \mathbf{R}_{\geqslant t_3}$, of $s' \in \mathbf{R}_{\geqslant 0}$ such that $d(\widetilde{\gamma}(t), \gamma'(s')) \leq H_0(\lambda, v)v(t)$. Moreover $s' = d(\widetilde{\gamma}(0), s) \geqslant d(\widetilde{\gamma}(0), \widetilde{\gamma}(t)) - H_0(\lambda, v)v(t) \geqslant \underline{\lambda}t - (1 + H_0(\lambda, v))\,v(t)$. Hence for all $t \in \mathbf{R}$ such that $t \geqslant t_4 := \sup\left\{t_3, r_{\underline{\lambda}/(2+2H_0(\lambda,v))}(v)\right\}$,

$$s' \geqslant t/(2\lambda). \tag{3.15}$$

Set $t_5 := t_4 \vee \sup\{r : v(r) \leqslant 8\delta\}$, $t_{\backsimeq}^0 := t_5 \vee 16\delta$ and then $t_{\backsimeq} = t_{\backsimeq}^0 + 2\lambda|\widetilde{\gamma}(0)|$. By (3.15), $s' \geqslant |\widetilde{\gamma}(0)| + 16\delta$. Moreover γ' and γ are asymptotic, so that by Lemma 3.3

on asymptotic geodesic rays, $d(\gamma'(s'), \mathrm{im}(\gamma)) \leqslant 8\delta$. By the triangle inequality and the definition of $t_{\simeq}$ we conclude that

$$d(\widetilde{\gamma}(t), \gamma) \leqslant |\widetilde{\gamma}(t) - \gamma'(s')| + d(\gamma'(s'), \mathrm{im}(\gamma)) \leqslant (1 + H_0(\lambda, v))\, v(t).$$

By construction, $t^0_{\simeq}$ only depends on λ, v, δ, so (3.4) is reached in the general case. From now on we proceed to attain (3.5). As before start by assuming $|\widetilde{\gamma}(0)| = 0$. For all $t \in \mathbf{R}_{\geqslant 0}$, since $\widetilde{\gamma}_{|[0,t]}$ is $v(t)$-connected, $p_\gamma(\widetilde{\gamma}_{|[0,t]})$ is $v(t) + 16\delta$-connected by Lemma 2.3, in particular it is $2v(t)$-connected as soon as $t \geqslant t_6 := \sup\{r : v(r) \leqslant 16\delta\}$. On the other hand, by Lemma 3.2, if $t \geqslant t_{\bigcirc}$ then $|\widetilde{\gamma}(t)| \geqslant (\underline{\lambda}/3)t$. Hence, if $t \geqslant t_7 := \sup\{t_6, t_{\bigcirc}\}$, the convex hull of $p_\gamma(\widetilde{\gamma}_{|[0,t]})$ contains $\gamma([0, (\underline{\lambda}/3)t - Hv(t)])$ where H is the constant from (3.4) (note that t_7 only depends on v, λ, δ since we are assuming $\widetilde{\gamma}(0) = o$).

Hence for all $t \geqslant t_8 = \sup\{t_7, r_{\underline{\lambda}/(6H)}(v)\}$, every $s \in [0, (\underline{\lambda}/6)t]$ lies between two orthogonal projections of points of $\widetilde{\gamma}_{|[0,t]}$ on γ. Define $R_8 := t_8/(6\lambda)$. For all $s \in \mathbf{R}$ such that $s \geqslant R_8$, there is $t_s \in [0, 6\lambda s]$ such that

$$|\gamma(s) - p_\gamma(\widetilde{\gamma}(t_s)| \leqslant 2v(6\lambda s) \leqslant 2(v \uparrow 6\lambda)v(s). \tag{3.16}$$

By the triangle inequality,

$$\begin{aligned} |\gamma(s) - \widetilde{\gamma}(t_s)| &\leqslant |\gamma(s) - p_\gamma(\widetilde{\gamma}(t_s))| + |p_\gamma(\widetilde{\gamma}(t_s)) - \widetilde{\gamma}(t_s)| \\ &\leqslant H_0 v(t_s) + 2(v \uparrow 6\lambda)v(s) \\ &\leqslant 2(v \uparrow 6\lambda)(H_0 + 1)v(s) \text{ for } s \geqslant R_8, \end{aligned} \tag{3.17}$$

where we used that $v(t_s) \leqslant (v \uparrow 6\lambda)v(s)$ for the last inequality. Set $\widetilde{H}_0(\lambda, v) := 2(v \uparrow 6\lambda)(H_0 + 1)$, and assume from now that $\widetilde{\gamma}(0)$ is arbitrary. Define $R^0_{\simeq} = R_8 \vee \sup\{r : \widetilde{H}_0 v(r) \leqslant 8\lambda\} \vee 16\delta$ and $\widetilde{H} = 2\widetilde{H}_0$. Then by Lemma 3.3 applied to $\gamma = o\eta$ and $\gamma' = \widetilde{\gamma}(0)\eta$, (3.17) and the triangle inequality, for all $s \geqslant R^0_{\simeq} + |\widetilde{\gamma}(0)|$, $d(\gamma(s), \mathrm{im}(\widetilde{\gamma})) \leqslant \widetilde{H} v(s)$. ☐

3.C. **Geodesics.** Our next aim consists in tracking $O(u)$-geodesics $\widetilde{\gamma}$. For this we need two steps:

(1) Control the Gromov product of ends $\partial_\infty \widetilde{\gamma}(-\infty)$ and $\partial_\infty \widetilde{\gamma}(+\infty)$ with respect to $|\widetilde{\gamma}(0)|$. This is achieved by Lemma 3.7.
(2) Track $\widetilde{\gamma}$ near both ends, starting at a distance linearly controlled by their Gromov product, and interpolate in between using the classical version of the stability lemma. This strategy is set up in Lemma 3.8.

Beware that, in contrast to the situation with (quasi)geodesics, one cannot reparametrize a (λ, v)-geodesic (e.g. to assume that $\widetilde{\gamma}(0)$ is the closest[10] point $\widetilde{b}$ to o in $\mathrm{im}(\widetilde{\gamma})$) without changing the function v. For this reason, and in order to simplify bounds on the tracking distance in step (2), we introduce an additional constant L and, from Lemma 3.8 on, make the assumption that $|\widetilde{\gamma}(0)| \leqslant L\widetilde{b}$.

Lemma 3.7. *Let $\delta \in \mathbf{R}_{\geqslant 0}$, $\lambda \in \mathbf{R}_{\geqslant 1}$ be constants, and let (Y, o) be a pointed proper geodesic δ-hyperbolic space. Let v be an admissible function. Let $\widetilde{\gamma}$ be a (λ, v)-geodesic into Y. Denote $\eta_\pm$ in $\partial_\infty X$ its endpoints, precisely $\eta_\pm = \partial_\infty \widetilde{\gamma}(\pm\infty)$. Then there exist $K = K(\lambda, v, \delta)$ and $R_\sqcap = R_\sqcap(\lambda, v, \delta)$, both in $\mathbf{R}_{>0}$ such that if $|\widetilde{\gamma}(0)| \geqslant R_\sqcap$,*

$$(\eta_- \mid \eta_+)_o \leqslant K|\widetilde{\gamma}(0)|. \tag{3.18}$$

[10]Such a point $\widetilde{b}$ exists since $\widetilde{\gamma}$ is proper.

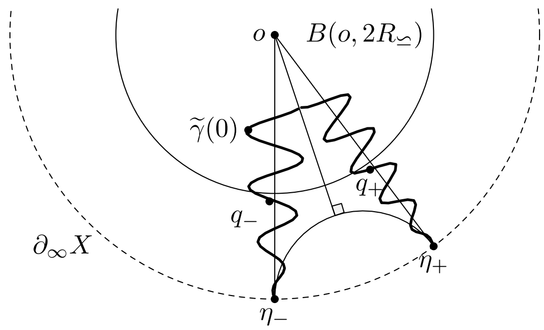


FIGURE 6. Main objects occurring in the proof of Lemma 3.7. The geodesic ray $\gamma_\pm$ from o to $\eta_\pm$ intersects the sphere $\partial B(o, 2R_{\backsimeq})$ at point $p_\pm$ (not depicted). Beware that the reasoning is by contradiction: this picture is not realistic.

Proof. The proof uses that $O(u)$-geodesics cannot make large round trips; see figure 6. Assume by contradiction that $(\eta_- \mid \eta_+)_o \geqslant 3(R^0_{\backsimeq} + |\widetilde{\gamma}(0)|) + 4\delta = 3R_{\backsimeq} + 4\delta$ for $\widetilde{\gamma}(0)$ arbitrarily far. Track the rays $\widetilde{\gamma}_- : t \mapsto \widetilde{\gamma}(-t)$ and $\widetilde{\gamma}_+ : t \mapsto \widetilde{\gamma}(t)$ with geodesic rays γ_- and γ_+. Let $\gamma = (\eta_- \eta_+)$ be a geodesic line. Define $p_\pm$ as the intersection point of $\gamma_\pm$ and $\partial B(o, 2R_{\backsimeq})$, i.e. $p_\pm = \gamma_\pm(2R_{\backsimeq})$. The twice-ideal triangle $o\eta_-\eta_+$ is 4δ-thin, and by the triangle inequality

$$\begin{aligned} d(p_\pm, \gamma) &\geqslant d(o, p_\gamma(o)) - d(o, p_\pm) \\ &\geqslant (\eta_- \mid \eta_+)_o - 2R_{\backsimeq} \geqslant R_{\backsimeq} + 4\delta > 4\delta, \end{aligned}$$

so $d(p_\pm, \gamma_\mp) < 4\delta$ and $|p_- - p_+| \leqslant 8\delta$ (where we used that both points p_+ and p_- lie on the same sphere centered at o). By sublinear tracking lemma 3.4, there is $q_\pm$ on $\operatorname{im}(\widetilde{\gamma}_\pm)$ such that $|p_\pm - q_\pm| \leqslant \widetilde{H} v(2R_{\backsimeq})$, and thanks to the triangle inequality,

$$|q_+ - q_-| \leqslant |p_+ - p_-| + 2\widetilde{H} v(2R_{\backsimeq}) \leqslant 8\delta + 2\widetilde{H} v(2R_{\backsimeq}). \tag{3.19}$$

Let t_+, t_- in $\mathbf{R}$ be such that $q_\pm = \widetilde{\gamma}(t_\pm)$, and write $T = \sup\{|t_+|, |t_-|\}$. The portion of $\widetilde{\gamma}$ between t_- and t_+ is a $(\lambda, v(T))$ quasi-geodesic segment. By a length-distance estimate for quasi-geodesics, for α large enough,

$$\begin{aligned} \ell_\alpha\left(\widetilde{\gamma}_{|[t_-, t_+]}\right) &\leqslant 2\lambda(t_+ - t_-) \leqslant 2\lambda\left(\lambda |q_- - q_+| + v(T)\right) \\ &\underset{(3.19)}{\leqslant} 4\lambda^2 H v(2R_{\backsimeq}) + 2\lambda v(T) + 16\lambda^2\delta. \end{aligned} \tag{3.20}$$

T can be bounded from above for $|\widetilde{\gamma}(0)|$ large enough:

$$\begin{aligned} \underline{\lambda} T - v(T) &\leqslant \sup\left\{|\widetilde{\gamma}(0) - q_-|, |\widetilde{\gamma}(0) - q_+|\right\} \\ &\leqslant 2R^0_{\backsimeq} + |\widetilde{\gamma}(0)| + 2\widetilde{H} v(T), \end{aligned}$$

so that since $v(T) \ll T$, there is a constant T_0 depending on v, λ (explicitly $T_0 = r_{1/(8\lambda\widetilde{H})}(v)$) such that $T \leqslant \inf\{T_0, \lambda(2R^0_{\backsimeq} + |\widetilde{\gamma}(0)|)\}$. On the other hand, $\ell_\alpha\left(\widetilde{\gamma}_{|[t_-, t_+]}\right)$ is greater than $|q_+ - \widetilde{\gamma}(0)| + |q_- - \widetilde{\gamma}(0)|$, and

$$\begin{aligned} |q_+ - \widetilde{\gamma}(0)| + |q_- - \widetilde{\gamma}(0)| &\geqslant |p_+ - \widetilde{\gamma}(0)| + |p_- - \widetilde{\gamma}(0)| - 2Hv(2R_{\backsimeq}) \\ &\geqslant 2R^0_{\backsimeq} + |\widetilde{\gamma}(0)| - 2Hv(2R_{\backsimeq}). \end{aligned}$$

Substitute this in (3.20) and make all dependences over $|\widetilde{\gamma}(0)|$ explicit:

$$\begin{aligned} 2R_{\backsimeq}^0 + |\widetilde{\gamma}(0)| - 2Hv(2R_{\backsimeq}) \leqslant & 4\lambda^2 Hv(2R_{\backsimeq}) + 2\lambda v(T) + 16\lambda^2\delta. \\ \leqslant & 4\lambda^2 Hv(2R_{\backsimeq}) + 2\lambda v(T_0) \\ & + 2\lambda v((\lambda/2)(2R_{\backsimeq})) + 16\lambda^2\delta. \end{aligned}$$

The last inequality rewrites under the form

$$\begin{aligned} |\widetilde{\gamma}(0)| &\leqslant \left[4\lambda^2 H + 2\lambda\,(v \uparrow \lambda)\right][v \uparrow 2]\, v(R_{\backsimeq}) + 2\lambda v(T_0) + 16\lambda^2\delta + 2R_{\backsimeq}^0 \\ &\leqslant H_3 v\left(R_{\backsimeq}^0 + |\widetilde{\gamma}(0)|\right) + \frac{\lambda}{4\widetilde{H}} r_{1/(8\lambda\widetilde{H})}(v) + 16\lambda^2\delta + 2R_{\backsimeq}^0, \end{aligned} \tag{3.21}$$

where $H_3 = \left[4\lambda^2 H + 2\lambda\,(v \uparrow \lambda)\right][v \uparrow 2]$. If $|\widetilde{\gamma}(0)| \geqslant 3R_{\backsimeq}^0$ then (3.21) yields

$$|\widetilde{\gamma}(0)| \leqslant 3(v \uparrow 2) H_3 v(|\widetilde{\gamma}(0)|) + \frac{3\lambda}{4\widetilde{H}} r_{1/(8\lambda\widetilde{H})}(v) + 48\lambda^2\delta.$$

This inequality would lead to a contradiction for $|\widetilde{\gamma}(0)|$ larger than

$$R_{\sqcap} := 3R_{\backsimeq}^0 \vee \left(\frac{3\lambda}{2\widetilde{H}} r_{1/(8\lambda\widetilde{H})}(v) + 96\lambda^2\delta\right) \vee r_{1/(6(v\uparrow 2)H_3(\lambda,\delta,v))}(v), \tag{3.22}$$

precisely, if $|\widetilde{\gamma}(0)| \geqslant R_{\sqcap}$, then $(\eta_- \mid \eta_+)_o \leqslant 3(R_{\backsimeq}^0 + |\widetilde{\gamma}(0)|) + 4\delta \leqslant 5|\widetilde{\gamma}(0)|$ as $R_{\sqcap} \geqslant R_{\backsimeq}^0 \vee 4\delta$. One may take $K = 5$. □

Lemma 3.8 (Tracking for $O(u)$-geodesics)**.** *Let $\delta \in \mathbf{R}_{\geqslant 0}$, $\lambda \in \mathbf{R}_{\geqslant 1}$, let u be an admissible function and let $v = O(u)$ be nondecreasing. Let (Y, o) be a proper geodesic pointed δ-hyperbolic space, and let $\widetilde{\gamma} : \mathbf{R} \to Y$ be a (λ, v)-geodesic. Define $\widetilde{b}$ as a closest point to o in $\mathrm{im}(\widetilde{\gamma})$. Let $L \in \mathbf{R}_{\geqslant 1}$ be a real constant and assume that the Gromov product $(\partial_\infty\widetilde{\gamma}(+\infty) \mid \partial_\infty\widetilde{\gamma}(-\infty))_o$ is larger than 60δ. There exist constants $\widetilde{R} = \widetilde{R}(\lambda, \delta, v, L)$, H_2 and $\widetilde{H}_2$ in $\mathbf{R}_{>0}$ (depending on λ, v and L) such that if*

$$\widetilde{R} \leqslant |\widetilde{\gamma}(0)| \leqslant L|\widetilde{b}|, \tag{3.23}$$

then for any geodesic $\gamma : \mathbf{R} \to Y$ with $[\gamma]_{\pm\infty} = \partial_\infty\widetilde{\gamma}(\pm\infty)$ and $\gamma(0) = p_\gamma o$,

$$\forall t \in \mathbf{R}, d(\widetilde{\gamma}(t), \gamma) \leqslant H_2 v(|\widetilde{\gamma}(t)|), \text{ and} \tag{3.24}$$

$$\forall s \in \mathbf{R}, d(\gamma(s), \mathrm{im}(\widetilde{\gamma})) \leqslant \widetilde{H}_2 v(|\gamma(s)|). \tag{3.25}$$

Proof. We divide the proof into 4 steps.

Step 1: Setting. — As before, write $\eta_\pm = \partial_\infty\widetilde{\gamma}(\pm\infty)$, cut $\widetilde{\gamma}$ in two (λ, v)-geodesic rays $\widetilde{\gamma}_\pm$ starting at $\widetilde{\gamma}(0)$, and track $\widetilde{\gamma}_\pm$ with geodesic rays $\gamma_\pm$. Let $\gamma = (\eta_- \eta_+)$, parametrized in such a way that $p_\gamma o = \gamma(0)$. Define $\widetilde{R}_0 = R_\sqcap$ and start assuming $|\widetilde{\gamma}(0)| \geqslant \widetilde{R}_0$. Let k be a real parameter whose value should be fixed later; only assume for now that $k \geqslant 2K + 1$, where K is the constant from Lemma 3.7. Define

$$p_\pm := \gamma_\pm\left(k\left[|\widetilde{\gamma}(0)| + R_{\backsimeq}^0\right] \vee 2(2\lambda + 1) r_{1/3\lambda}(v) \vee r_{1/(2\widetilde{H})}(v)\right), \tag{3.26}$$

where $R_{\backsimeq}^0$ is the constant from Lemma 3.4 applied to $\widetilde{\gamma}_+$ or $\widetilde{\gamma}_-$ and let $q_\pm$ be a closest point to $p_\pm$ on $\mathrm{im}(\widetilde{\gamma}_\pm)$. Since $k \geqslant 2K$ and $|\widetilde{\gamma}(0)| \geqslant R_\sqcap$, by Lemma 3.7, $k|\widetilde{\gamma}(0)| \geqslant 2(\eta_- \mid \eta_+)_o$, and $|p_+| = |p_-| \geqslant 2(\eta_- \mid \eta_+)_o$ as well. Let σ be

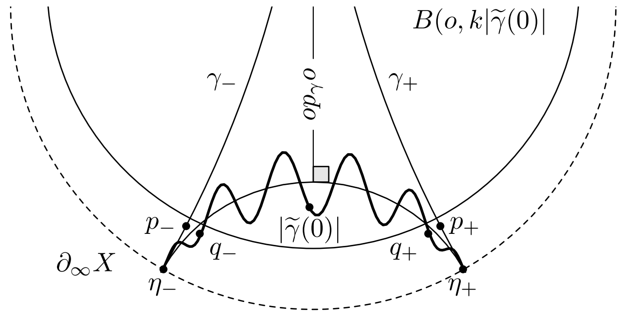


FIGURE 7. Main objects of proof of Lemma 3.8. Tracking is achieved by the classical Morse Lemma 2.9 between q_- and q_+ and by the ray tracking Lemma 3.4 beyond those points.

a geodesic segment from o to $p_\gamma o$. By the triangle inequality, $d(p_\pm, \sigma) \geqslant 2(\eta_- \mid \eta_+)_o - \sup_{c \in \sigma} |c| \geqslant (\eta_- \mid \eta_+)_o - 56\delta$. As $(\eta_- \mid \eta_+)_o > 60\delta$ by hypothesis,

$$d(p_\pm, \sigma) > 60\delta - 56\delta = 4\delta, \text{ hence } d(p_\pm, \gamma) \leqslant 4\delta, \tag{3.27}$$

where we used that the once-ideal right-angled triangles $o\eta_\pm(p_\gamma o)$ are 4δ-slim (recall that $p_\pm$ lies on the side $\gamma_\pm$ by definition). Further, because $k \geqslant 1$, inequality (3.5) of the tracking lemma 3.4 allows to bound $|q_- - p_-|$ and $|q_+ - p_+|$:

$$|p_\pm - q_\pm| \leqslant \widetilde{H} v(|p_\pm|), \tag{3.28}$$

so that by the triangle inequality and the definition (3.26) of $p_\pm$,

$$|q_\pm| \geqslant |p_\pm| - \widetilde{H} v(|p_\pm|) \underset{(3.26)}{\geqslant} \frac{1}{2}|p_\pm|. \tag{3.29}$$

Step 2: Selection of k. — At this point, in order to control the quasi-geodesic additive error term of $\widetilde{\gamma}$ between q_- and q_+ we need to select k large enough so that $|p_\pm| \geqslant R_\bigcirc$, where $R_\bigcirc$ is associated to $\widetilde{\gamma}$ as in Lemma 3.2. Recall from the expression (3.2) of $R_\bigcirc$ that $R_\bigcirc = 4|\widetilde{\gamma}(0)| \vee 2(2\lambda+1)(3\lambda|\widetilde{\gamma}(0)| \vee r_{1/(3\lambda)} v)$. Thus from now on we fix $k = (2K+1) \vee 8 \vee 12\lambda(2\lambda+1)$. By inequality (3.28), this is sufficient to ensure $|q_\pm| \geqslant R_\bigcirc$, and then using the estimates and notation of Lemma 3.2, the portion of $\widetilde{\gamma}$ situated between q_+ and q_- is a (λ, c)-quasigeodesic segment, with $c = \widehat{v}(|q_+| \vee |q_-|)$.

Step 3: Tracking between q_- and q_+, and estimation of H_2. — Let $\overline{\gamma}$ be a geodesic segment between q_+ and q_-. Let $t_\pm \in \mathbf{R}$ be such that $\widetilde{\gamma}(t_\pm) = \widetilde{\gamma}_\pm(\pm t_\pm) = q_\pm$. By Lemma 2.9 $\operatorname{dist}_{\mathrm{H}}(\overline{\gamma}, \widetilde{\gamma}_{|[t_-,t_+]}) \leqslant (h(\lambda) \vee \widetilde{h}(\lambda))(\delta + c)$, and by hyperbolic geometry, letting $s_\pm$ be such that $\gamma(s_\pm) = p_\gamma(q_\pm)$, $\operatorname{dist}_{\mathrm{H}}(\overline{\gamma}, \gamma_{|[s_-,s_+]})$ cannot be much greater than the pairwise distance between the endpoints of these geodesic segments:

$$\begin{aligned} \operatorname{dist}_{\mathrm{H}}(\overline{\gamma}, \gamma_{|[s_-,s_+]}) &\leqslant |q_\pm - \gamma(s_\pm)| + 8\delta \\ &\leqslant 4\delta + \widetilde{H} v(|p_\pm|) + 8\delta, \end{aligned} \tag{3.30}$$

where we combined (3.27) and (3.28) by means of the triangle inequality. Hence

$$\begin{aligned}\forall t\in[t_-,t_+],\ d(\widetilde{\gamma}(t),\gamma) &\leqslant (h(\lambda)\vee\widetilde{h}(\lambda))(\delta+(v\uparrow 3\lambda)v(|q_+|\vee|q_-|))\\ &\quad +8\delta+4\delta+\widetilde{H}v\left(|q_+|\vee|q_-|\right)\\ &\leqslant \left(12+\widetilde{h}(\lambda)\right)\delta+(\widetilde{h}(\lambda)(v\uparrow 3\lambda)+\widetilde{H})v\left(|p_\pm|+\widetilde{H}v(|p_\pm|\right).\end{aligned}\tag{3.31}$$

(here $\widetilde{h}(\lambda)$ is used alone as it is equal to $\widetilde{h}(\lambda)\vee h(\lambda)$). If $|\widetilde{\gamma}(0)|\geqslant R_\sqcap$, then in view of the definition of $p_\pm$ (3.26),

$$\begin{aligned}|p_\pm| &= k\left[|\widetilde{\gamma}(0)|+R^0_{\backsimeq}\right]\vee 2(2\lambda+1)r_{1/3\lambda}(v)\vee r_{1/(2\widetilde{H})}(v)\\ &\leqslant \left(k+\frac{R^0_{\backsimeq}}{R_\sqcap}\right)|\widetilde{\gamma}(0)|\vee 2(2\lambda+1)r_{1/3\lambda}(v)\vee r_{1/(2\widetilde{H})}(v)\\ &\underset{(3.22)}{\leqslant} (2k|\widetilde{\gamma}(0)|)\vee 2(2\lambda+1)r_{1/3\lambda}(v)\vee r_{1/(2\widetilde{H})}(v)\end{aligned}\tag{3.32}$$

where we used $k\geqslant 1$ so that $k+1/3\leqslant 2k$ in the last inequality. Define $\widetilde{R}_1=\widetilde{R}_0\vee 2(2\lambda+1)r_{1/3\lambda}(v)\vee r_{1/(2\widetilde{H})}(v)$. By (3.32), if $|\widetilde{\gamma}(0)|\geqslant\widetilde{R}_1$,

$$v\left(|p_\pm|+\widetilde{H}v(|p_\pm|)\right)\leqslant v\left(\frac{3}{2}|p_\pm|\right)\leqslant v(3k|\widetilde{\gamma}(0)|).$$

Recall that by the right-hand side of assumption (3.23), $|\widetilde{\gamma}(0)|\leqslant L|\widetilde{b}|=L\inf\{|\widetilde{\gamma}(t)|: t\in\mathbf{R}\}$. Plugging (3.32) in (3.31), one obtains that for all t in $[t_-,t_+]$,

$$\begin{aligned}d(\widetilde{\gamma}(t),\gamma) &\leqslant (12+\widetilde{h}(\lambda))\delta+\left(\widetilde{h}(\lambda)(v\uparrow 3\lambda)+\widetilde{H}\right)v(3k|\widetilde{\gamma}(0)|)\\ &\leqslant (12+\widetilde{h}(\lambda))\delta+2\widetilde{H}(v\uparrow 3\lambda)v(3Lk|\widetilde{\gamma}(t)|),\end{aligned}$$

where we used that $\widetilde{H}\geqslant\widetilde{h}(\lambda)$ on the second line; this is because by definition, $\widetilde{H}=4(v\uparrow 6\lambda)(2h(\lambda)(v\uparrow 1{+}8\lambda^2){+}1)\geqslant 8h(\lambda)\geqslant\widetilde{h}(\lambda)$. Define $\widetilde{R}_3=\widetilde{R}_2\vee L^{-1}\sup\{r: v(r)\leqslant(12+\widetilde{h}(\lambda))\delta\}$. If $|\widetilde{\gamma}(0)|\geqslant\widetilde{R}_3$, the right-hand side of assumption (3.23) ensures that $v(|\widetilde{\gamma}(t)|)\geqslant(12+\widetilde{h}(\lambda))\delta$ for all t, so we have proved

$$\forall t\in[t_-,t_+],\ d(\widetilde{\gamma}(t),\gamma)\leqslant(1+2(v\uparrow 3\lambda)\widetilde{H}(v\uparrow 3Lk))v(|\widetilde{\gamma}(t)|).\tag{3.33}$$

On the other hand, in view of the tracking lemma 3.4, for all $t\in(-\infty,t_-)$, $d(\widetilde{\gamma}(t),\gamma_-)\leqslant(v\uparrow 3\lambda)Hv(|\widetilde{\gamma}(t)|)$ and similarly for all $t\in(t_+,+\infty)$, $d(\widetilde{\gamma}(t),\gamma_+)\leqslant(v\uparrow 3\lambda)v(|\widetilde{\gamma}(t)|)$. Since the twice-ideal triangle $o\eta_-\eta_+$ is 4δ-slim, using the triangle inequality and the fact that $v(|\widetilde{\gamma}(t)|)\geqslant 12\delta$ for all t provided $|\widetilde{\gamma}(0)|\geqslant\widetilde{R}_3$ by definition of $\widetilde{R}_3$,

$$\forall t\in\mathbf{R}\setminus[t_-,t_+],\ d(\widetilde{\gamma}(t),\gamma)\leqslant((v\uparrow 3\lambda)H+4\delta/(12\delta))v(|\widetilde{\gamma}(t)|).\tag{3.34}$$

Putting (3.33) and (3.34) together yields the expected tracking inequality (3.24) for the provisional $\widetilde{R}_3$. Precisely H_2 may be taken as

$$H_2=4(v\uparrow 3Lk)(v\uparrow 3\lambda)\widetilde{H}\vee 2(v\uparrow 3\lambda)H.\tag{3.35}$$

Step 4: Estimation of the tracking constant $\widetilde{H}_2$. — From here, one could deduce (3.25) using (3.24) for $\widetilde{R}$ large enough by a metric connectedness argument as in Lemma 2.9 or Lemma 3.4, but let us rather use the former estimates from the current proof. Define $\widetilde{R}_4=\widetilde{R}_3\vee r_{1/(2LH_2)}(v)$; then if $|\widetilde{\gamma}(0)|\geqslant\widetilde{R}_4$, it follows from the tracking inequality just obtained for $\widetilde{\gamma}$ that $|p_\gamma o| = |\gamma(0)| \geqslant (1/2)|\widetilde{\gamma}(0)|$. Then for all $s \in [s_-, s_+]$, by (3.30),

$$d(\gamma(s), \overline{\gamma}) \leqslant \widetilde{H} v(|p_\pm|) + 12\delta \leqslant 2\widetilde{H} v(2k|\gamma(s)|) \vee 24\delta. \tag{3.36}$$

On the other hand, recall that by Lemma 2.9, for all $c \in \overline{\gamma}$, $d(c, \widetilde{\gamma} \leqslant \widetilde{h}(\lambda)(\delta + (v \uparrow 3\lambda) v(|q_+| \vee |q_-|)$. Combining this with (3.36) by means of the triangle inequality while remembering the bound on $|q_\pm|$ implied by (3.28), one obtains

$$\forall s \in [s_-, s_+],\ d(\gamma(s), \widetilde{\gamma}) \leqslant (2\widetilde{H} + \widetilde{h}(\lambda)(\delta + (v \uparrow 3\lambda)) v(3k|\gamma(s)|). \tag{3.37}$$

Finally, if $s \in \mathbf{R}$ is such that $s \leqslant s_-$ or $s \geqslant s_+$, since o, $p_\gamma o$ and $\gamma(s)$ are 28δ-almost lined up, $|\gamma(s)| \geqslant |s| - |p_\gamma o| \geqslant |s|/2$. $\gamma(s)$ is at most 4δ away from its orthogonal projection on $\gamma_{\epsilon(s)}$, where $\epsilon(s)$ is the sign of s. Given the definition of $p_\pm$, $p_{\gamma_{\epsilon(s)}} \gamma(s)$ is at a distance at least $R_{\backsimeq}$ from the origin, and inequality (3.5) from Lemma 3.4 bounds its distance to $\widetilde{\gamma}$ so that

$$\begin{aligned} d(\gamma(s), \widetilde{\gamma}) &\leqslant |\gamma(s) - p_{\gamma_{\epsilon(s)}} \gamma(s)| + d(p_{\gamma_{\epsilon(s)}} \gamma(s), \widetilde{\gamma}) \\ &\leqslant 4\delta + \widetilde{H} v(|p_{\gamma_{\epsilon(s)}} \gamma(s)|) \leqslant 2\widetilde{H} v(|\gamma(s)|). \end{aligned}$$

Together with (3.37), this proves (3.25) with $\widetilde{R} = \widetilde{R}_4$ and

$$\widetilde{H}_2 = \left(2\widetilde{H} + \widetilde{h}(\lambda)\right)(\delta + (v \uparrow 3\lambda))(v \uparrow 3k). \tag{3.38}$$

□

3.D. Distance between $O(u)$-geodesics.

Lemma 3.9. *Let $\delta \in \mathbf{R}_{\geqslant 0}$ be a constant. Let γ_1 and γ_2 be geodesic lines into a δ-hyperbolic space, with four pairwise distinct endpoints. Define $\Delta = d(\mathrm{im}(\gamma_1), \mathrm{im}(\gamma_2))$. Then for all $s_1, s_2 \in \mathbf{R}$,*

$$|\gamma_1(s_1) - \gamma_2(s_2)| \geqslant \Delta + d(\gamma_1(s_1), p_{\gamma_1}\mathrm{im}(\gamma_2)) \vee d(\gamma_2(s_2), p_{\gamma_2}\mathrm{im}(\gamma_1)) - 56\delta. \tag{3.39}$$

Proof. The distance on the left is symmetric relatively to $\gamma_i(s_i)$, so it suffices to prove $|\gamma_1(s_1) - \gamma_2(s_2)| \geqslant \Delta + d(\gamma_1(s_1), p_{\gamma_1}\mathrm{im}(\gamma_2)) - 56\delta$. The points $\gamma_1(s_1), p_{\gamma_2}(\gamma_1(s_1))$ and $\gamma_2(s_2)$ are the vertices of a right-angled hyperbolic triangle so that by Lemma 2.6, they are 28δ-almost lined up. By the triangle inequality,

$$\begin{aligned} d(\gamma_1(s_1), \gamma_2(s_2)) + 2 \cdot 28\delta &\geqslant d(\gamma_1(s_1), p_{\gamma_2}(\gamma_1(s_1))) + d(p_{\gamma_2}(\gamma_1(s_1)), \gamma_2(s_2)) \\ &\geqslant \Delta + d(\gamma_1(s_1), p_{\gamma_1}\mathrm{im}(\gamma_2)). \end{aligned}$$

□

Lemma 3.10. *Let v^1 and v^2 be admissible functions, and define $v = v^1 \vee v^2$. Let $L \in \mathbf{R}_{>1}$ be a constant. Let δ be a hyperbolicity constant and let $\lambda = (\underline{\lambda}, \overline{\lambda}) \in \mathbf{R}^2_{>0}$ be expansion and Lipschitz constants. There exist $J = J(\lambda, v, L)$, $R = R(\delta, \lambda, v, L)$ and, for $i \in \{1, 2\}$, $\widetilde{R}^i = \widetilde{R}^i(\delta, \lambda, v^i, L)$ in $\mathbf{R}_{>0}$ such that for any proper geodesic, pointed δ-hyperbolic space (Y, o), if $(\gamma_1, \widetilde{\gamma}_1)$ and $(\gamma_2, \widetilde{\gamma}_2)$ are such that*

(i) *γ_1, γ_2 are geodesics $\mathbf{R} \to Y$ with four distinct endpoints $\eta_i^\pm = \gamma_i(\pm\infty)$,*
(ii) *for $i \in \{1, 2\}$, $\widetilde{\gamma}_i$, is a (λ, v^i)-geodesics $\mathbf{R} \to Y$,*
(iii) *for $i \in \{1, 2\}$, $\partial_\infty \widetilde{\gamma}_i(\pm\infty) = [\gamma_i]_\pm$,*
(iv) *$\boxtimes\{\eta_1^\pm, \eta_2^\pm\} \geqslant 60\delta$, and $\overline{\boxtimes}\{\eta_1^\pm, \eta_2^\pm\} \geqslant R$,*
(v) *for all $i \in \{1, 2\}$, $\widetilde{R}^i \leqslant |\widetilde{\gamma}_i(0)| \leqslant L \inf_{t \in \mathbf{R}} |\widetilde{\gamma}_i(t)|$,*

then

$$|d(\gamma_1, \gamma_2) - d(\widetilde{\gamma}_1, \widetilde{\gamma}_2)| \leqslant J v\left(\overline{\boxtimes}\{\eta_1^\pm, \eta_2^\pm\}\right). \tag{3.40}$$

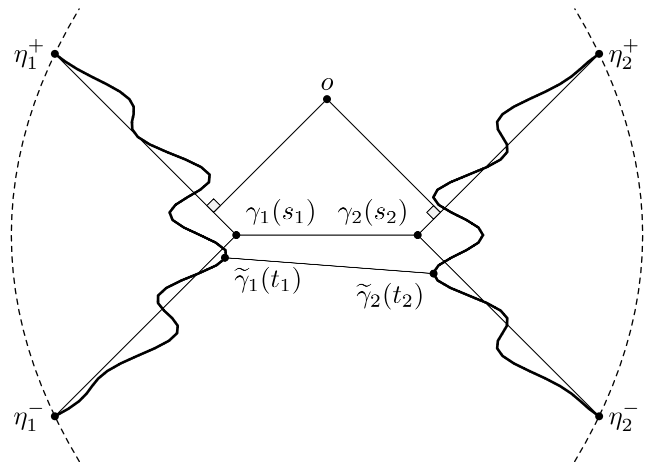


FIGURE 8. Main points occuring in the proof of Lemma 3.10. Straight, resp. wavy lines depict geodesic, resp. $O(u)$-geodesic lines; the boundary is dashed.

Sketch of proof for Lemma 3.10. See figure 8. The main tool is the geodesic tracking lemma 3.8; however the tracking between $\widetilde{\gamma}_i$ and γ_i becomes inefficient far from the origin. Thus we need to prove that shortest geodesic segments between γ_1 and γ_2 on the one hand, and between $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ on the other hand, are close to the origin (at most not significantly farther than the largest Gromov product). The part concerning γ_1 and γ_2 was already expressed by Lemma 2.8; as for the other part we show (inequality (3.44)) that letting t_1, t_2 be such that $d(\widetilde{\gamma}_1, \widetilde{\gamma}_2) = |\widetilde{\gamma}_1(t_1) - \widetilde{\gamma}_2(t_2)|$, $|\widetilde{\gamma}_1(t_1)| \vee |\widetilde{\gamma}_2(t_2)|$ is linearly controlled by $\boxtimes\{\eta_1^\pm, \eta_2^\pm\}$ on the large-scale. This uses the well-known behavior described by Lemma 3.9: geodesic rays spread apart linearly from each other after the Gromov products are reached; since they track $O(u)$-geodesics at a distance growing sublinearly, $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ also spread away from each other, which prevents $\widetilde{\gamma}_i(t_i)$ from being much farther than all the Gromov products.

Proof. For $i \in \{1, 2\}$, let $s_i \in \mathbf{R}$ be such that $|\gamma_1(s_1) - \gamma_2(s_2)| = d(\gamma_1, \gamma_2)$. As $\gamma_1(s_1) \in p_{\gamma_1}(\gamma_2)$, and similarly $\gamma_2(s_2) \in p_{\gamma_2}(\gamma_1)$, by the projection lemma 2.8, $\sup_i |\gamma_i(s_i)| \leqslant \boxtimes\left\{\eta_1^\pm, \eta_2^\pm\right\} + 284\delta$. Further, set $\widetilde{R}^i = \widetilde{R}^i(\lambda, \delta, v^i, L)$ according to the tracking lemma 3.8. Note that by the assumptions (i) to (iii), the first inequality in assumption (iv) and the right-hand side inequality in assumption (v), applied to the pairs $(\gamma_i, \widetilde{\gamma}_i)$, by Lemma 3.8,

$$\begin{aligned} \forall i \in \{1,2\},\ d(\gamma_i(s_i), \widetilde{\gamma}_i) &\leqslant \widetilde{H}_2 v(|\gamma_i(s_i)|) \\ &\leqslant \widetilde{H}_2 v(\boxtimes\left\{\eta_1^\pm, \eta_2^\pm\right\} + 284\delta). \end{aligned}$$

By the triangle inequality, setting $J^+ = 2\widetilde{H}_2(v \uparrow 2)$ and $R_0 = \sup\{r : v(r) \leqslant 284\delta\}$, as soon as $\boxtimes\left\{\eta_1^\pm, \eta_2^\pm\right\} \geqslant R_0$,

$$d(\widetilde{\gamma}_1, \widetilde{\gamma}_2) - d(\gamma_1, \gamma_2) \leqslant d(\gamma_1(s_1), \widetilde{\gamma}_1) + d(\gamma_2(s_2), \widetilde{\gamma}_2) \leqslant J^+ v\left(\boxtimes\left\{\eta_1^\pm, \eta_2^\pm\right\}\right). \tag{3.41}$$

This is one half of inequality (3.40).

For $i \in \{1,2\}$ let $t_i \in \mathbf{R}$ be such that $d(\widetilde{\gamma}_1, \widetilde{\gamma}_2) = |\widetilde{\gamma}_1(t_1) - \widetilde{\gamma}_2(t_2)|$. Let $\widetilde{s}_i$ be such that $\gamma_i(\widetilde{s}_i) = p_{\gamma_i}\widetilde{\gamma}_i(t_i)$. By the triangle inequality and the tracking lemma 3.8,

$$|\gamma_1(\widetilde{s}_1) - \gamma_2(\widetilde{s}_2)| \leqslant d(\widetilde{\gamma}_1, \widetilde{\gamma}_2) + 2H_2 v(|\widetilde{\gamma}_1(t_1)| \vee |\widetilde{\gamma}_2(t_2)|). \tag{3.42}$$

Inequality (3.39) of Lemma 3.9 gives a lower bound on $|\gamma_1(\widetilde{s}_1) - \gamma_2(\widetilde{s}_2)|$, which can be plugged into (3.42) yielding

$$\begin{aligned} d(\gamma_1, \gamma_2) + & d(\gamma_1(\widetilde{s}_1), p_{\gamma_1}\mathrm{im}(\gamma_2)) \vee d(\gamma_2(\widetilde{s}_2), p_{\gamma_2}\mathrm{im}(\gamma_1)) \\ & \leqslant d(\widetilde{\gamma}_1, \widetilde{\gamma}_2) + 2H_2 v(|\widetilde{\gamma}_1(t_1)| \vee |\widetilde{\gamma}_2(t_2)|) + 56\delta. \end{aligned} \tag{3.43}$$

On the other hand, using twice the triangle inequality and Lemma 2.8,

$$\begin{aligned} d(\gamma_1(\widetilde{s}_1), p_{\gamma_1}\mathrm{im}(\gamma_2)) \vee d(\gamma_2(\widetilde{s}_2), p_{\gamma_2}\mathrm{im}(\gamma_1)) & \geqslant |\gamma_1(\widetilde{s}_1)| \vee |\gamma_1(\widetilde{s}_1)| \\ & \quad - \overline{\boxtimes}\left\{\eta_1^\pm, \eta_2^\pm\right\} - 284\delta \\ & \geqslant |\widetilde{\gamma}_1(t_1)| \vee |\widetilde{\gamma}_2(t_2)| - \overline{\boxtimes}\left\{\eta_1^\pm, \eta_2^\pm\right\} \\ & \quad - H_2 v(|\widetilde{\gamma}_1(t_1)| \vee |\widetilde{\gamma}_2(t_2)|) - 284\delta. \end{aligned}$$

Reorganizing (3.43),

$$\begin{aligned} |\widetilde{\gamma}_1(t_1)| \vee |\widetilde{\gamma}_2(t_2)| & \leqslant \overline{\boxtimes}\{\eta_1^\pm, \eta_2^\pm\} + 340\delta + 3H_2 v(|\widetilde{\gamma}_1(t_1)| \vee |\widetilde{\gamma}_2(t_2)|) \\ & \quad + d(\widetilde{\gamma}_1, \widetilde{\gamma}_2) - d(\gamma_1, \gamma_2) \\ & \underset{(3.41)}{\leqslant} \overline{\boxtimes}\{\eta_1^\pm, \eta_2^\pm\} + 340\delta + 3H_2 v(|\widetilde{\gamma}_1(t_1)| \vee |\widetilde{\gamma}_2(t_2)|) \\ & \quad + J^+ v\left(\overline{\boxtimes}\{\eta_1^\pm, \eta_2^\pm\}\right) \end{aligned}$$

when $\overline{\boxtimes}\left\{\eta_1^\pm, \eta_2^\pm\right\} \geqslant F_0$. Hence,

$$\begin{aligned} |\widetilde{\gamma}_1(t_1)| \vee |\widetilde{\gamma}_2(t_2)| \leqslant \inf \big\{ & r_{1/(6H_2)}(v), \\ & 2\left[\overline{\boxtimes}\{\eta_1^\pm, \eta_2^\pm\} + 340\delta + J^+ v\left(\overline{\boxtimes}\{\eta_1^\pm, \eta_2^\pm\}\right)\right]\big\}. \end{aligned}$$

Set $R_1 = \sup\{r : v(r) \geqslant 584\delta/J^+\}$ and $R_2 = \sup\{R_0, R_1, r_{1/(2J^+)}(v)\}$. Then if $\overline{\boxtimes}\left\{\eta_1^\pm, \eta_2^\pm\right\} \geqslant R_2$,

$$|\widetilde{\gamma}_1(t_1)| \vee |\widetilde{\gamma}_2(t_2)| \leqslant \inf\left\{r_{1/(6H_2)}(v), 4\overline{\boxtimes}\{\eta_1^\pm, \eta_2^\pm\}\right\}. \tag{3.44}$$

Thus if $R_3 = r_{1/(4H_2)}(v)$, and if $t_1, t_2 \in \mathbf{R}$ are such that $|\widetilde{\gamma}_1(t_1)| \vee |\widetilde{\gamma}_2(t_2)| \geqslant R_3$, then

$$H_2 v(\widetilde{\gamma}_1(t_1)| \vee |\widetilde{\gamma}_2(t_2)|) \leqslant H_2 (v \uparrow 4) v\left(\overline{\boxtimes}\{\eta_1^\pm, \eta_2^\pm\}\right).$$

Finally by the triangle inequality, writing $J^- = 2H_2(v \uparrow 4)$,

$$\begin{aligned} d(\gamma_1, \gamma_2) - d(\widetilde{\gamma}_1, \widetilde{\gamma}_2) & \leqslant d(\gamma_1(\widetilde{s}_1), \gamma_2(\widetilde{s}_2)) - d(\widetilde{\gamma}_1(t_1), \widetilde{\gamma}_2(t_2)) \\ & \leqslant J^- v\left(\overline{\boxtimes}\{\eta_1^\pm, \eta_2^\pm\}\right). \end{aligned} \tag{3.45}$$

To reach the conclusion of Lemma 3.10, define $J = J^- \vee J^+$ and then combine (3.41) with (3.45). $\square$

3.E. **Tracking radii.** While there are four relevant parameters (λ, v, δ, L) to express $R^0_{\unrhd}$, $R_\sqcap$, $\widetilde{R}$ and R, only the dependence on v is of interest for what follows. Consequently, a constant depending on the remaining parameters λ, δ, L can be written as, e.g., $C(\lambda, \delta)$ or $C(\lambda, \delta, L)$.

Lemma 3.11. *Let v be an admissible function. Let $\lambda \in \mathbf{R}_{\geqslant 1}$ be a biLipschitz constant. Let δ be a hyperbolicity constant. There exist a positive integer n and constants $C(\lambda)$, $C(\lambda,\delta)$, $C(\lambda,\delta,L)$ such that in Lemma 3.4, Lemma 3.7, Lemma 3.8 and Lemma 3.10, the tracking radii may be taken as*

$$R^0_{\backsimeq} = r_{C(\lambda)(v\uparrow 1+\lambda)^{-n}}(v) \vee C(\lambda,\delta)\left(1+\sup\{r : v(r) \leqslant C(\lambda,\delta)\}\right) \tag{3.46}$$

$$\widetilde{R} = C(\lambda,\delta) r_{C(\lambda)(v\uparrow L)^{-1}(v\uparrow 1+\lambda)^{-n}}(v) \vee \left(1+\sup\{r : v(r) \leqslant C(\lambda,\delta,L)\}\right) \tag{3.47}$$

$$R = r_{C(\lambda,\delta)(v\uparrow 1+\lambda)^{-n}}(v) \vee C(\lambda,\delta)\left(1+\sup\{r : v(r) \leqslant C(\lambda,\delta,L)\}\right) \tag{3.48}$$

Proof. It will be used without further notice that $r_\alpha(v) \vee r_\beta(v) = r_{\alpha\wedge\beta}(v)$, for all $\alpha,\beta \in \mathbf{R}_{>0}$, and that $\lambda \geqslant 1$, especially $1/\lambda \leqslant \lambda \leqslant \lambda^2$. The bounds we obtain need not be excessively precise, and we allow losing multiplicative factors frequently. Start with (3.46), and notation as in the proof of Lemma 3.4.

$$\begin{aligned} t_1 &= r_{2\underline{\lambda}}(v) \vee r_{1/(3\lambda)}(v) \vee 3\lambda r_{1/(2h(\lambda)v\uparrow 1+8\lambda^2)}(v) \vee \sup\{s : v(s) \leqslant \delta\} \\ &\quad \vee 12\lambda\delta \vee (4\lambda M'(\underline{\lambda},\delta)+1) \vee \sup\{r : v(r) \leqslant 6\lambda^2\delta\} \\ &\leqslant 3\lambda r_{1/(2h(\lambda)v\uparrow 1+8\lambda^2)}(v) \vee C(\lambda,\delta). \end{aligned} \tag{3.49}$$

Next, $t_3 = t_1 \vee \sup\{r : v(r) \leqslant h(\lambda)\delta\}$ since $h(\lambda)\delta \geqslant 6\lambda^2\delta$. After that, $t \geqslant t_4 := \sup\left\{t_3, r_{\underline{\lambda}/(2+2H_0(\lambda,v))}(v)\right\}$, where $H_0(\lambda,v) = 2h(\lambda)(v \uparrow 1+8\lambda^2)+1$. From this and (3.49) we deduce

$$t_4 \leqslant r_{1/(3\lambda H_0)}(v) \vee \sup\{r : v(r) \leqslant h(\lambda)\delta\} \vee C(\lambda,\delta) \tag{3.50}$$

and then

$$\begin{aligned} t_5 &\leqslant t_4 \vee \sup\{r : v(r) \leqslant 8\delta\} \\ &\leqslant r_{1/(3\lambda H_0)}(v) \vee \sup\{r : v(r) \leqslant 8h(\lambda)\delta\} \vee C(\lambda,\delta); \end{aligned}$$

$$\begin{aligned} t_{\backsimeq^0} &\leqslant t_5 \vee 16\delta = t_5 \vee C(\lambda,\delta); \\ t_6 &= t_{\backsimeq^0} \vee \sup\{r : v(r) \leqslant 8\delta\} \\ &\leqslant r_{1/(3\lambda H_0)}(v) \vee \sup\{r : v(r) \leqslant 16h(\lambda)\delta\} \vee C(\lambda,\delta). \end{aligned}$$

As $t_7 = t_6 \vee t_{\bigcirc}$, $t_{\bigcirc} = r_{1/(3\lambda)}(v)$ and $H_0 \geqslant 1$, the same bound applies to t_7. Next,

$$t_8 = t_7 \vee r_{\underline{\lambda}/(6H)}(v) \leqslant r_{\underline{\lambda}/(6H)}(v) \vee \sup\{r : v(r) \leqslant 16h(\lambda)\delta\} \vee C(\lambda,\delta) \tag{3.51}$$

(remember that $H = 1+H_0$ by definition). Thus

$$\begin{aligned} R^0_{\backsimeq} &= t_8/(6\lambda) \vee \sup\{r : 2(v\uparrow 6\lambda)(H_0+1)v(r) \leqslant 8\lambda\} \\ &\leqslant t_8 \vee \sup\{r : 2(H_0+1)v(6\lambda r) \leqslant 8\lambda\} \\ &\leqslant t_8 \vee \sup\left\{r : 2\left(2h(\lambda)(v\uparrow 1+8\lambda^2)+1\right)v(6\lambda r) \leqslant 8\lambda\right\} \\ &\leqslant t_8 \vee \sup\left\{r : 6h(\lambda)v(6\lambda(1+8\lambda^2)r) \leqslant 8\lambda\right\} \\ &\underset{(3.51)}{\leqslant} r_{1/(6\lambda H)}(v) \vee C(\lambda,\delta)\left(1+\sup\{r : v(r) \leqslant C(\lambda,\delta)\}\right). \end{aligned} \tag{3.52}$$

This inequality implies (3.46) (one may take $n=4$ there), since $H = 2h(\lambda)(v \uparrow 1+8\lambda^2)+2 \leqslant C(\lambda)(v\uparrow 1+\lambda)^4$. Let us turn to (3.47). Start establishing a similar bound for $R_{\sqcap}$, with notation as in the proof of Lemma 3.7. By (3.22),

$$R_{\sqcap} = 3R^0_{\backsimeq} \vee \frac{3\lambda}{2\widetilde{H}} r_{1/(8\lambda\widetilde{H})}(v) \vee 192\lambda^2\delta \vee r_{1/(6(v\uparrow 2)H_3(\lambda,v,\delta))}(v),$$

where $H_3 = \left(4\lambda^2 H + 2\lambda(v \uparrow \lambda)\right)(v \uparrow 2) \leqslant 6\lambda^2 H(v\uparrow 2) \leqslant 6\lambda^2 \widetilde{H}$, hence by (3.52),

$$R_\sqcap \leqslant 3\lambda r_{1/(8\lambda^2 \widetilde{H}(v\uparrow 2))}(v) \vee C(\lambda,\delta)\left(1+\sup\left\{r : v(r) \leqslant C(\lambda,\delta)\right\}\right). \tag{3.53}$$

In the proof of Lemma 3.8, $\widetilde{R}$ was defined as a supremum of four terms:

$$\begin{aligned}\widetilde{R} &= R_\sqcap \vee 2(2\lambda+1) r_{1/(3\lambda)}(v) \vee r_{1/(2\widetilde{H})}(v) \vee r_{1/(2LH_2)}(v)\\ &\quad \vee L^{-1}\sup\{r : v(r) \leqslant (12+\widetilde{h}(\lambda))\delta\}\\ &\leqslant R_\sqcap \vee 5\lambda^2 r_{1/(3\lambda)\wedge 1/(2\widetilde{H})\wedge 1/(2LH_2)}(v) \vee C(\lambda,\delta).\end{aligned}$$

We need to bound the tracking constants $\widetilde{H}$ and H_2. By definition of $\widetilde{H}$ in the proof of Lemma 3.4, $\widetilde{H} = 4(v\uparrow 6\lambda)\left(2h(\lambda)(v\uparrow 1+8\lambda^2)+1\right) \leqslant C(\lambda)\,(v\uparrow 1+\lambda)^{n_0}$, where n_0 is large enough, and by (3.35) with $k = 11 \vee 12\lambda(2\lambda+1) \leqslant 36\lambda^2$,

$$H_2 \leqslant 4\left[(v\uparrow 3Lk)(v\uparrow 3\lambda)\widetilde{H}\right] \vee 2(v\uparrow 3\lambda)H \leqslant C(\lambda)(v\uparrow L)(v\uparrow 1+\lambda)^{n_1},$$

where n_1 is large enough. By (3.53) and the previous bounds, $\widetilde{R}$ may be taken as

$$\widetilde{R} = 5\lambda^2 r_{1/(C(\lambda)(v\uparrow L)(v\uparrow 1+\lambda)^{n_0})} \vee C(\lambda,\delta)(1+\sup\{r : v(r) \leqslant C(\lambda,\delta)\}).$$

This is a precise form of (3.47). Finally, we must prove (3.48). With notation as in the proof of Lemma 3.10, $R = r_{1/(2J^+)}(v) \vee \sup\{r : v(r) \leqslant 284\delta \vee 584\delta/J^+\}$, where $J^+ = 2\widetilde{H}_2(v\uparrow 2)$ so that

$$R \leqslant r_{1/(4\widetilde{H}_2(v\uparrow 2))}(v) \vee \sup\{r : v(2r) \leqslant 584\delta/(2\widetilde{H}_2)\}, \tag{3.54}$$

and we need to bound $\widetilde{H}_2$. With notation as in the proof of Lemma 3.8, recall from (3.35) that $\widetilde{H}_2$ can be bounded by

$$\begin{aligned}\widetilde{H}_2 = (2\widetilde{H}+\widetilde{h}(\lambda))(\delta+(v\uparrow 3\lambda)) &\leqslant C(\lambda)\widetilde{H}C(\lambda,\delta)(v\uparrow 3\lambda)(v\uparrow 3k)\\ &\leqslant C(\lambda,\delta)(v\uparrow 1+\lambda)^7(v\uparrow 3\lambda)(v\uparrow 3\cdot 36\lambda^2)(v\uparrow 3\lambda).\end{aligned}$$

Plugging this inequality in (3.54) yields the expected (3.48). □

Lemma 3.12 (Sublinear growth of tracking radii)**.** *Let w be an admissible function. For all $p \in \mathbf{R}_{\geqslant 0}$, define $w_p(r) = w(p+r)$, and then denote by R_p, resp. $\widetilde{R}_p$ the constants $R(\lambda,\delta,w_p)$ and $\widetilde{R}(\lambda,\delta,w_p)$ of Lemma 3.10. There exist $\widetilde{K} = \widetilde{K}(\lambda,\delta,w,L)$ and $K = K(\lambda,\delta,w,L)$ in $\mathbf{R}_{>0}$ such that*

$$\widetilde{R}_p \leqslant \widetilde{K}w(p), \text{ and} \tag{3.55}$$

$$R_p \leqslant Kw(p). \tag{3.56}$$

Proof. By Lemma 3.11, there exists a positive integer n such that $\widetilde{R}_p$ and R_p may be taken as

$$\begin{aligned}\widetilde{R}_p &= C(\lambda,\delta) r_{C(\lambda)(w_p\uparrow L)^{-1}(w_p\uparrow 1+\lambda)^{-n}}(w_p)\\ &\quad \vee C(\lambda,\delta)\left(1+\sup\left\{r : w_p(r) \leqslant C(\lambda,\delta,L)\right\}\right)\end{aligned} \tag{3.57}$$

$$R_p = r_{C(\lambda)(w_p\uparrow 1+\lambda)^{-n}}(w_p) \vee C(\lambda,\delta)\left(1+\sup\left\{r : w_p(r) \leqslant C(\lambda,\delta,L)\right\}\right). \tag{3.58}$$

The rightmost terms $C(\lambda,\delta)\left(1+\sup\left\{r : w_p(r) \leqslant C(\lambda,\delta,L)\right\}\right)$ are nonincreasing functions of p, since $\{w_p\}$ is a nondecreasing sequence of functions, so that their dependence over p can be removed. Further, $w_p \uparrow 1+\lambda$ is a nonincreasing function of p by Lemma 1.1 (1), hence $(w_p \uparrow 1+\lambda)^{-n}$ is a nondecreasing function of p. Thus (3.57) and (3.58) may be simplified as

$$\widetilde{R}_p = C(\lambda,\delta) r_{C(\lambda)(w_p\uparrow L)^{-1}(w\uparrow 1+\lambda)^{-n}}(w_p) \vee C(\lambda,\delta,L,w)$$
$$R_p = r_{C(\lambda)(w\uparrow 1+\lambda)^{-n}}(w_p) \vee C(\lambda,\delta,L,w).$$

Then by Lemma 1.1 (2), $\widetilde{R}_p \leqslant C(\lambda,\delta,L,w) \vee C(\lambda)(w\uparrow 2)(w\uparrow 1+\lambda)^n w(p)$. This proves (3.55) for a constant $\widetilde{K} = \widetilde{K}(\lambda,\delta,L,w)$, and similarly there exists $K = K(\lambda,\delta,L,w)$ such that $R_p \leqslant Kw(p)$, which is (3.56). □

4. On the sphere at infinity

4.A. **Sublinearly quasiMöbius homeomorphisms.** With geodesic boundaries of hyperbolic spaces in mind, we abstractly define sublinearly quasiMöbius homeomorphisms between compact metric spaces:

Definition 4.1. Let u be an admissible function. Let $(\underline{\alpha},\overline{\alpha}) \in \mathbf{R}^2_{>0}$ be a couple of constants. Let (Ξ,ϱ) and (Ψ,ϑ) be metric spaces and let $\varphi : \Xi \to \Psi$ be a homeomorphism. φ is a $(\underline{\alpha},\overline{\alpha},O(u))$-sublinearly quasiMöbius homeomorphism if there exist $v = O(u)$, $\nu \in \mathbf{R}_{>1}$ and $\mathcal{E} \in \mathbf{R}_{>0}$ such that for all $(\xi_1,\dots\xi_4) \in \Xi^4$ with $0 < \inf_{i\neq j} \varrho(\xi_i,\xi_j) \leqslant \sup_{i\neq j} \varrho(\xi_i,\xi_j) < \mathcal{E}$,

$$\underline{\alpha}\log^+_\nu[\xi_i] - v\left(\sup_{i\neq j}\left[-\log_\nu \varrho(\xi_i,\xi_j)\right]\right) \leqslant \log^+_\nu\left[\varphi(\xi_i)\right]$$
$$\overline{\alpha}\log^+_\nu[\xi_i] + v\left(\sup_{i\neq j}\left[-\log_\nu \varrho(\xi_i,\xi_j)\right]\right) \geqslant \log^+_\nu\left[\varphi(\xi_i)\right].$$

Note that one would only need a change of function v within the $O(u)$-class to compensate a different choice of ν. We call $\underline{\alpha}$, $\overline{\alpha}$ and $\alpha = \sup\{\overline{\alpha}, 1/\underline{\alpha}\}$ the Lipschitz-Möbius constants of φ.

Although this is not a direct consequence of Definition 4.1, sublinearly quasi-Möbius homeomorphisms between uniformly perfect spaces are stable under composition; we postpone the proof to subsection 4.B. Also note that in the definition one could replace the source and target distances with any equivalent real-valued kernels $\widehat{\varrho}$ and $\widehat{\vartheta}$, or even, if no special attention is required on precise Lipschitz-Möbius constants, with kernels such that $\widehat{\varrho}^{\gamma_1}$ and $\widehat{\vartheta}^{\gamma_2}$ are equivalent to ϱ and ϑ for a pair of exponents $\gamma_1,\gamma_2 \in \mathbf{R}_{>0}$. This occurs on geodesic boundaries when $\widehat{\varrho}$ and $\widehat{\vartheta}$ are visual quasimetrics while ϱ and ϑ are visual distances.

Recall that, by Proposition 1.5, any large-scale sublinearly Lipschitz embedding f between proper geodesic Gromov-hyperbolic spaces induces a boundary map, which only depends on the $O(u)$-closeness class of f so that it can be denoted $\partial_\infty[f]_{O(u)}$.

Theorem 4.2. *Let u be an admissible function. Let $(\underline{\lambda},\overline{\lambda}) \in \mathbf{R}^2_{>0}$ be expansion and Lipschitz constants. Let $f : X \to Y$ be a $(\underline{\lambda},\overline{\lambda},O(u))$-sublinearly biLipschitz equivalence betwen proper, geodesic hyperbolic spaces. Then $\partial_\infty[f]_{O(u)}$ is a $(\underline{\lambda},\overline{\lambda},O(u))$-sublinearly quasiMöbius homeomorphism.*

Sketch of proof for Theorem 4.2. Our argument is inspired from the lecture notes by Bourdon [3, Theorem 2.2] on Mostow rigidity and Tukia's theorem; the main ingredient is Lemma 3.10, which ensures that the geometric interpretation of the cross-difference (see Proposition 1.9 and Figure 4) subsists with a sublinear error when applying a sublinearly biLipschitz equivalence and measuring distances between $O(u)$-geodesics in the target space. Lemma 3.10 must be applied with care, though, since the control functions and tracking radii deteriorate as the Gromov products of endpoints grow. This is where Lemma 3.12 intervenes and certifies that the growth of tracking radii is sublinear with respect to Gromov products, so that the tracking estimates and their consequences are ultimately valid.

Proof. Fix basepoints o in X and Y, and let $w = O(u)$ be an admissible function such that f is a $(\underline{\lambda}, \overline{\lambda}, w)$-sublinearly biLipschitz equivalence from (X, o) to (Y, o). For any quadruple $(\xi_1, \dots \xi_4) \in \partial_\infty^4 X$, write for short $\eta_i = \partial f(\xi_i)$ for all i in $\{1, \dots 4\}$, and for all $\varepsilon, \mathcal{E} \in \mathbf{R}_{>0}$ such that $\varepsilon < \mathcal{E}$, let $F(\varepsilon, \mathcal{E})$ be the subspace of $\partial^4 X$ defined by

$$\begin{cases} \overline{\boxtimes}\{\xi_i\} > -\log_\mu \varepsilon, \\ \underline{\boxtimes}\{\xi_i\} > -\log_\mu \mathcal{E}. \end{cases}$$

Note that, since $\partial_\infty X$ is compact the space defined by the first inequality is a neighborhood of the ends in $\partial_\infty^4 X$, hence it suffices to prove the inequality

$$\underline{\lambda}[\xi_i] - v\left(\overline{\boxtimes}\{\xi_i\}\right) \leqslant [\eta_i] \leqslant \overline{\lambda}[\xi_i] + v\left(\overline{\boxtimes}\{\xi_i\}\right)$$

for all $(\xi_i) \in F(\varepsilon, \mathcal{E})$, for some small ε and $\mathcal{E}$ and $v = O(u)$. For any pair $\{i, j\} \in \{\{1,4\},\{2,3\}\}$ let χ_{ij} be a geodesic in X with endpoints ξ_i and ξ_j, resp. γ_{ij} a geodesic in Y with endpoints η_i and η_j such that $\chi_{ij}(0) = p_{\chi_{ij}}(o)$ and $\gamma_{ij}(0) = p_{\gamma_{ij}}(o)$. Finally, write $\widetilde{\gamma}_{ij}(t) = f \circ \chi_{ij}(t)$, and observe that $\widetilde{\gamma}_{ij}$ is a (λ, w')-geodesic, where $w'(r) := w(|\chi_{14}(r)| \vee |\chi_{23}(r)|) \leqslant w\left((\xi_1 \mid \xi_4)_o \vee (\xi_2 \mid \xi_3)_o + r\right)$. Especially, $\widetilde{\gamma}_{ij}$ is a (λ, w_p) geodesic, where

$$w_p(r) := w(p + r),$$

and $p = (\xi_i \mid \xi_j)_o$. We shall apply Lemma 3.10 with $v^1 = w_{(\xi_1|\xi_4)}$, $v^2 = w_{(\xi_2|\xi_3)}$ and $v = w_{\overline{\boxtimes}\{\xi_i\}}$. Assumptions (i), (ii) and (iii) follow from the definitions of γ_{ij} and $\widetilde{\gamma}_{ij}$. Then, recall from inequality (3.1) in Lemma 3.2 that if $|\chi_{ij}(0)| \geqslant t_\bigcirc(|f(o)|, w)$, then for all $t \in \mathbf{R}$, $\frac{1}{3\lambda}|\chi_{ij}(t)| \leqslant |\widetilde{\gamma}_{ij}(t)| \leqslant 3\lambda|\chi_{ij}(t)|$, and then

$$\forall t \in \mathbf{R},\ |\widetilde{\gamma_{ij}}(0)| \leqslant 3\lambda|\chi_{ij}(0)| \leqslant 3\lambda|\chi_{ij}(t)| \leqslant 9\lambda^2|\widetilde{\gamma_{ij}}(t)|.$$

This is the right-hand side inequality of (v) with $L = 9\lambda^2$, that we fix for the rest of the proof. Observe that the lower bound needed on the radii $|\chi_{ij}(0)|$ is guaranteed as soon as $\underline{\boxtimes}\{\xi_i\} \geqslant t_\bigcirc(|f(o)|, w) = r_{1/(3\lambda)}(w) \vee 3\lambda|f(o)|$. On the other hand, by Cornulier's theorem 1.7 $\partial_\infty[f]$ is uniformly continuous on $\partial_\infty X$, so there exists $R_\square \in \mathbf{R}_{\geqslant 0}$ such that $\underline{\boxtimes}\{\xi_i\} \geqslant R_\square \implies \underline{\boxtimes}\{\eta_i\} \geqslant 60\delta$. Let $\widetilde{K} = \widetilde{K}(\lambda, w, \delta, L)$ be the constant from Lemma 3.12, and define

$$\mathcal{E} = \mu^{-(R_\square \vee t_\bigcirc(|f(o)|, w) \vee r_{1/(3\lambda\widetilde{K})}(w))}.$$

Then as soon as $\underline{\boxtimes}\{\xi_i\} > -\log_\mu \mathcal{E}$,

$$\begin{cases} \underline{\boxtimes}\{\eta_i\} \geqslant 60\delta. & \text{as } \underline{\boxtimes}\{\xi_i\} \geqslant R_\square \\ |\widetilde{\gamma}_{ij}(0)| \leqslant L \inf_{t \in \mathbf{R}} |\widetilde{\gamma_{ij}}(t)| & \text{as } \underline{\boxtimes}\{\xi_i\} \geqslant t_\bigcirc(|f(o)|, w) \\ \widetilde{R}_{(\xi_i|\xi_j)_o} \leqslant \frac{(\xi_i|\xi_j)_o}{3\lambda} \leqslant |\widetilde{\gamma}_i(0)| & \text{as } (\xi_i \mid \xi_j)_o \geqslant \underline{\boxtimes}\{\xi_i\} \geqslant t_\bigcirc(|f(o)|, w) \vee r_{1/(3\lambda\widetilde{K})}(w). \end{cases}$$

The first line is the first condition in (iv), the second and third one are the assumption (v); we used (3.55) from Lemma 3.12 in the third line. By the conclusion of Cornulier's theorem 1.7 applied to both $\partial_\infty[f]$ and to $\partial_\infty[f]^{-1}$, there exists $\varepsilon_0 \in \mathbf{R}_{>0}$ such that

$$\overline{\boxtimes}\{\xi_i\} > -\log_\mu \varepsilon_0 \implies 2\lambda\overline{\boxtimes}\{\xi_i\} \geqslant \overline{\boxtimes}\{\eta_i\} \geqslant \frac{1}{2\lambda}\overline{\boxtimes}\{\xi_i\}.$$

Let K be the constant from Lemma 3.12. Define $\varepsilon = \varepsilon_0 \wedge \mathcal{E} \wedge \mu^{-2\lambda r_{(1/3\lambda K)}(w)}$. Then by (3.56) of Lemma 3.12, $\overline{\boxtimes}\{\xi_i\} > -\log_\mu \varepsilon \implies \overline{\boxtimes}\{\eta_i\} \geqslant R_{\overline{\boxtimes}\{\xi_i\}}$. Thus if $(\xi_i) \in F(\varepsilon, \mathcal{E})$ then Lemma 3.10 applies to $(\gamma_{ij}, \widetilde{\gamma}_{ij})$, and

$$\begin{aligned} |d_Y(\gamma_{23}, \gamma_{14}) - d_Y(\widetilde{\gamma}_{23}, \widetilde{\gamma}_{14})| &\leqslant J w_{\overline{\boxtimes}\{\xi_i\}}(\overline{\boxtimes}\{\eta_i\}) \\ &\leqslant J(w \uparrow 2\lambda) w(\overline{\boxtimes}\{\eta_i\}). \end{aligned} \tag{4.1}$$

Thanks to Proposition 1.9, there exists $C = C(\delta)$ in $\mathbf{R}_{\geqslant 0}$ such that

$$\begin{cases} d_X(\chi_{14}, \chi_{23}) - C(\delta) & \leqslant \log^+[\xi_i] \leqslant d_X(\chi_{14}, \chi_{23}) + C(\delta). \\ d_Y(\gamma_{14}, \gamma_{23}) - C(\delta) & \leqslant \log^+[\eta_i] \leqslant d_Y(\gamma_{14}, \gamma_{23}) + C(\delta). \end{cases}$$

In view of (4.1) and the previous set of inequalities, it suffices to prove

$$\underline{\lambda} d_X(\chi_{14}, \chi_{23}) - v\left(\overline{\boxtimes}\{\xi_i\}\right) \leqslant d_Y(\gamma_{14}, \gamma_{23}) \leqslant \overline{\lambda} d_X(\chi_{14}, \chi_{23}) + v\left(\overline{\boxtimes}\{\xi_i\}\right) \tag{4.2}$$

for some function $v = O(u)$. Start with the left-hand side inequality. Letting $\widetilde{s}_1, \widetilde{s}_2 \in \mathbf{R}$ be such that $|f \circ \chi_{14}(\widetilde{s}_1) - f \circ \chi_{23}(\widetilde{s}_2)| = d(\widetilde{\gamma}_{14}, \widetilde{\gamma}_{23})$,

$$\begin{aligned} \underline{\lambda} d(\chi_{14}, \chi_{23}) - w\left(\overline{\boxtimes}\{\xi_i\}\right) &\leqslant \underline{\lambda}|\chi_{14}(\widetilde{s}_1) - \chi_{23}(\widetilde{s}_2)| - w\left(|\chi_{14}(\widetilde{s}_1)| \vee |\chi_{23}(\widetilde{s}_2)|\right) \\ &\leqslant |f \circ \chi_{14}(\widetilde{s}_1) - f \circ \chi_{23}(\widetilde{s}_2)| \\ &= d(\widetilde{\gamma}_{14}, \widetilde{\gamma}_{23}) \\ &\underset{(4.1)}{\leqslant} d(\gamma_{14}, \gamma_{23}) + J(w \uparrow 2\lambda)\left(\overline{\boxtimes}\{\eta_i\}\right), \end{aligned}$$

hence

$$\underline{\lambda} d(\chi_{14}, \chi_{23}) \leqslant d(\gamma_{14}, \gamma_{23}) + (1 + J(w \uparrow 2\lambda))\, w\left(\overline{\boxtimes}\{\eta_i\} + \overline{\boxtimes}\{\xi_i\}\right). \tag{4.3}$$

Let us proceed in the same way for the right-hand side of (4.2). By Lemma 3.10, letting $s_1, s_2 \in \mathbf{R}$ be such that $|\chi_{14}(s_1) - \chi_{23}(s_2)| = d(\chi_{14}, \chi_{23})$,

$$\begin{aligned} d(\gamma_{14}, \gamma_{23}) &\leqslant d(\widetilde{\gamma}_{14}, \widetilde{\gamma}_{23}) + J(w \uparrow 2\lambda)\left(\overline{\boxtimes}\{\eta_i\}\right) \\ &\leqslant |\widetilde{\gamma}_{14}(s_1) - \widetilde{\gamma}_{23}(s_2)| + J(w \uparrow 2\lambda)\left(\overline{\boxtimes}\{\eta_i\}\right) \\ &\leqslant \overline{\lambda} d(\chi_{14}, \chi_{23}) + \left(1 + (w \uparrow 2\lambda)^2\right) w\left(\overline{\boxtimes}\{\xi_i\}\right). \end{aligned} \tag{4.4}$$

Setting $v = \left(1 + (w \uparrow 2\lambda)^2\right) w$ this proves (4.2) and the theorem. □

4.B. **Properties of sublinearly quasiMöbius homeomorphisms.** After simplifying the cross-ratio estimates when two, resp. one points are far away, one obtains that sublinearly quasiMöbius homeomorphisms between appropriate spaces are Hölder, resp. almost quasisymmetric, see figure 9. Precisely we work under the following assumption (Buyalo and Schroeder [5, 7.2] or Mackay and Tyson [23, 1.3.2]); see however Remark 4.8.

Definition 4.3. Let Ξ be a metric space. Then X is uniformly perfect if there exists $\tau \in (0, 1)$ such that for every ball $B \subsetneq \Xi$, the annulus $B \setminus \tau B$ is non-empty.

Note that in the definition, for any positive integer k, up to replacing τ with τ^k one can assume for free that $B \setminus \tau B$ has k points. Uniform perfectness is granted for boundaries of non-elementary hyperbolic groups, or for connected spaces.

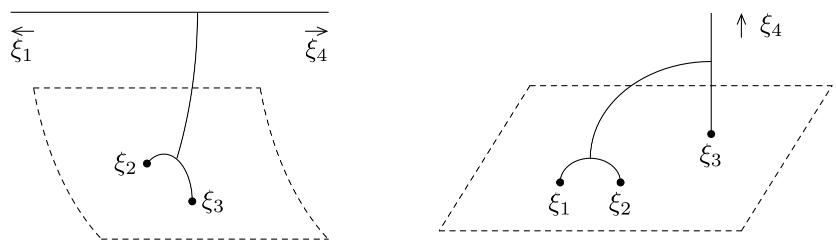


FIGURE 9. Hyperbolic ideal tetrahedra. On the left two points are far away from the remaining pair; on the right one point is far from the remaining triple.

Proposition 4.4 ("almost" Hölder continuity)**.** *Let* $(\Xi; \varrho)$ *and* (Ψ, ϑ) *be compact uniformly perfect metric spaces and let* $\varphi : \Xi \to \Psi$ *be a* $(\underline{\lambda}, \overline{\lambda}, O(u))$*-sublinearly quasiMöbius homeomorphism. Then* φ *admits a modulus of continuity*

$$\omega(t) = \exp\left(\underline{\lambda} \log t + v(-\log t)\right), \tag{4.5}$$

with $v = O(u)$*.*

Remark 4.5. As a consequence, under the same assumptions and for all $\alpha \in (0, \underline{\lambda})$, φ is α-Hölder continuous. Thus Theorem 4.2 may be seen as a strengthening of Cornulier's theorem (with the restriction made on spaces).

Proof. Let $\mathcal{E}$ be the constant from Definition 4.1 associated to φ, and let τ be such that Ξ is τ-uniformly perfect. Define

$$\mathcal{D}_1 := \frac{\tau^4}{4}\left(\mathcal{E} \wedge \frac{\operatorname{diam}\Xi}{3}\right) \text{ and}$$
$$\mathcal{D}_1' = \inf\left\{\vartheta(\varphi(\xi_1), \varphi(\xi_2)) : \xi_1, \xi_2 \in \Xi, \varrho(\varphi(\xi_1), \varphi(\xi_2)) \geqslant \left(\tau^{-1} - 1\right)\mathcal{D}_1\right\}.$$

Let ξ_1 and ξ_2 in Ξ be such that $\varrho(\xi_1, \xi_2) < \mathcal{D}_1$. The ball $B := \tau^{-4}B(\xi_1, \mathcal{D}_1)$ is not equal to Ξ (this would indeed contradict the definition of $\mathcal{D}_1$), so there exists $\alpha \in B \setminus \tau B$ and $\beta \in \tau^2 B \setminus \tau^3 B$. By the triangle inequality

$$\varrho(\alpha, \beta) \geqslant \left(\tau^{-3} - \tau^{-2}\right)\mathcal{D}_1 \text{ and } \varrho(\beta, \xi_2) \geqslant \left(\tau^{-1} - 1\right)\mathcal{D}_1,$$

for short

$$\inf_i \varrho(\alpha, \xi_i) \wedge \inf_i \varrho(\beta, \xi_i) \geqslant \left(\tau^{-1} - 1\right)\mathcal{D}_1. \tag{4.6}$$

Further, by definition of $\mathcal{D}_1'$, a similar inequality holds in the target space:

$$\inf_i \vartheta(\varphi(\alpha), \varphi(\xi_i)) \wedge \inf_i \vartheta(\varphi(\beta), \varphi(\xi_i)) \geqslant \mathcal{D}_1'. \tag{4.7}$$

By definition of the metric cross ratios,

$$\frac{\left(\tau^{-1} - 1\right)^2 \mathcal{D}_1^2}{\operatorname{diam}(\Xi)} \frac{1}{\varrho(\xi_1, \xi_2)} \leqslant [\alpha, \xi_1, \xi_2, \beta] \leqslant \frac{\operatorname{diam}(\Xi)^2}{\mathcal{D}_1} \frac{1}{\varrho(\xi_1, \xi_2)}.$$

$$\frac{\mathcal{D}_1'^2}{\operatorname{diam}(\Psi)} \frac{1}{\vartheta(\varphi(\xi_1), \varphi(\xi_2))} \leqslant [\alpha', \varphi(\xi_1), \varphi(\xi_2); \beta'] \leqslant \frac{\operatorname{diam}(\Psi)^2}{\mathcal{D}_1'} \frac{1}{\vartheta(\varphi(\xi_1), \varphi(\xi_2))}.$$

thus $\log^+[\alpha,\xi_1,\xi_2,\beta] - \log^+ \frac{1}{\varrho(\xi_1,\xi_2)}$ and $[\alpha',\varphi(\xi_1),\varphi(\xi_2);\beta'] - \log^+ \frac{1}{\vartheta(\varphi(\xi_1),\varphi(\xi_2))}$ are bounded by

$$\mathcal{L} = 2\left(\left|\log\frac{1-\tau}{\tau}\right| + |\log \mathcal{D}_1| + |\log \operatorname{diam}(\Xi)|\right) \text{ and } \mathcal{L}' = (|\log \mathcal{D}'| + |\log \operatorname{diam}(\Psi)|)$$

respectively. Now by hypothesis φ is $(\underline{\lambda},\overline{\lambda},v_0)$-sublinearly quasiMöbius for some $v_0 = O(u)$. By definition, setting $v = v_0 + \mathcal{L}$, for all ξ_1,ξ_2 such that $\varrho(\xi_1,\xi_2) < \mathcal{D}_1 \wedge 1$ (note that $\{\xi_1,\xi_2\}$ is the closest pair among ξ_1,ξ_2,α,β),

$$-\log\vartheta(\varphi(\xi_1),\varphi(\xi_2)) \leqslant \overline{\lambda}(-\log\varrho(\xi_1,\xi_2)) + v(-\log\varrho(\xi_1,\xi_2)), \tag{4.8}$$

$$-\log\vartheta(\varphi(\xi_1),\varphi(\xi_2)) \geqslant \underline{\lambda}(-\log\varrho(\xi_1,\xi_2)) - v(-\log\varrho(\xi_1,\xi_2)). \tag{4.9}$$

In particular the conclusion (4.5) is equivalent to the second inequality. □

The Hölder continuity (4.9) intervenes in the following analog of Lemma 3.2, a technical refinement of definition 4.1.

Lemma 4.6. *Let u be an admissible function. Let $(\underline{\alpha},\overline{\alpha})$ be Lipschitz-Möbius data. Let φ be a $(\underline{\alpha},\overline{\alpha},O(u))$ sublinearly quasiMöbius homeomorphism between compact uniformly perfect spaces $(\Xi,\widehat{\varrho})$ and $(\Psi,\widehat{\vartheta})$. There exist $\widehat{v} = O(u)$ and $\mathcal{E}_2 \in \mathbf{R}_{>0}$ such that for all $(\xi_1,\dots\xi_4) \in \Xi^4$ with $0 < \inf_{i\neq j}\varrho(\xi_i,\xi_j) \leqslant \sup_{i\neq j}\varrho(\xi_i,\xi_j) < \mathcal{E}_2$,*

$$\underline{\alpha}\log_\nu^+[\xi_i] - \widehat{v}\left(\sup_{i\neq j}\left[-\log_\nu\widehat{\varrho}(\xi_i,\xi_j)\right] \wedge \sup_{i\neq j}\left[-\log_\nu\widehat{\vartheta}(\varphi(\xi_i),\varphi(\xi_j))\right]\right) \leqslant \log_\nu^+[\varphi(\xi_i)]$$

$$\overline{\alpha}\log_\nu^+[\xi_i] + \widehat{v}\left(\sup_{i\neq j}\left[-\log_\nu\widehat{\varrho}(\xi_i,\xi_j)\right] \wedge \sup_{i\neq j}\left[-\log_\nu\widehat{\vartheta}(\varphi(\xi_i),\varphi(\xi_j))\right]\right) \geqslant \log_\nu^+[\varphi(\xi_i)],$$

where $\widehat{v} = O(u)$.

Proof. Let $v = O(u)$ be such that φ is a $(\underline{\alpha},\overline{\alpha},v)$-sublinearly quasiMöbius homeomorphism. Then by Proposition 4.4 and the fact that v is sublinear, there is $\mathcal{E}_H \in \mathbf{R}_{>0}$ such that for all $(\xi_1,\dots\xi_4) \in \Xi$ distinct and such that $\inf\widehat{\varrho}(\xi_1,\xi_2) \leqslant \mathcal{E}_H \wedge e^{-(\log\nu)r_{\underline{\lambda}/2}(v)}$,

$$\sup_{i\neq j}\left[-\log_\nu\widehat{\vartheta}(\varphi(\xi_i),\varphi(\xi_j))\right] \geqslant (\underline{\lambda}/2)\sup_{i\neq j}\left[-\log_\nu\widehat{\varrho}(\xi_i,\xi_j)\right].$$

The conclusion follows, with $\widehat{v} = \left(v \uparrow \frac{2}{\underline{\lambda}}\right)v$. □

Proposition 4.7. *Let u be an admissible function. $O(u)$-sublinearly quasiMöbius homeomorphisms form a groupoid $\mathcal{M}_{O(u)}$ with uniformly perfect compact metric spaces as objects. Composition in $\mathcal{M}_{O(u)}$ has a multiplicative effect on Lipschitz-Möbius and reverse Lipschitz-Möbius constants.*

Proof. Let Ω,Ξ,Ψ be compact metric spaces, and let $\varphi:\Xi\to\Psi$ and $\psi:\Omega\to\Xi$ be $O(u)$-quasiMöbius homeomorphisms, with respective parameters $(\underline{\alpha}_\varphi,\overline{\alpha}_\varphi,v_\varphi)$ and $(\underline{\alpha}_\psi,\overline{\alpha}_\psi,v_\psi)$. Let $(\omega_1,\dots,\omega_4)$ be a 4-tuple of distinct points in Ω ; for $i\in\{1,\dots,4\}$ set $\xi_i = \psi(\omega_i)$ and $\eta_i = \varphi(\xi_i)$. Set

$$w = \overline{\alpha}_\psi v_\varphi + \left(v \uparrow \frac{2}{\alpha_\varphi}\right)v_\psi.$$

Then by the previous lemma, $\varphi\circ\psi$ is a $(\underline{\alpha}_\varphi\underline{\alpha}_\psi,\overline{\alpha}_\varphi\overline{\alpha}_\psi,w)$-sublinearly quasiMöbius homeomorphism. □

Remark 4.8. The assumption of uniform perfectness (Definition 4.3) could be dropped in Proposition 4.7 if one adopts the heavier form of Definition 4.1 given by the inequalities of Lemma 4.6. It follows from the proof of Theorem 4.2 that this more restrictive definition is still valid for boundary maps of sublinearly biLipschitz equivalences.

We now turn to the scale-sensitive moduli distortion property of sublinearly quasiMöbius homeomorphisms. Recall that for any ξ in a metric space Ξ, the annulus $A = B(\xi, s) \setminus B(\xi, r)$ is said to have a modulus $\mathfrak{M} = \log(s/r)$.

Proposition 4.9. *Let φ be a $(\underline{\lambda}, \overline{\lambda}, O(u))$-sublinearly quasiMöbius homeomorphism between spaces (Ξ, ϱ) and (Ψ, ϑ). Assume that Ξ is uniformly perfect. There exist $\mathcal{D}_1 \in \mathbf{R}_{>0}$ and $w = O(u)$ such that the following holds: let $A \subset \Xi$ be an annulus of inner radius $r \in \mathbf{R}_{>0}$ and outer radius $R \in (r, \mathcal{D}_1]$. Then $\varphi(A)$ is contained in an annulus of modulus $2\lambda\mathfrak{M} + w(-\log r))$.*

Proof. Define $\mathcal{D}_1$ and $\mathcal{D}_1'$ as in the proof of Proposition 4.4. For any triple $(\xi_1, \xi_2, \xi_3) \in \Xi^3$ such that $\{\xi_1, \xi_2, \xi_3\}$ has diameter less than $\mathcal{D}_1$ one can find $\omega \in \tau^{-4} B(\xi_1, \mathcal{D}_1) \setminus \tau^{-3} B(\xi_1, \mathcal{D}_1)$. Define $\omega' = \varphi(\omega)$. By the triangle inequality and the definition of $\mathcal{D}_1'$

$$\begin{cases} \inf_i \varrho(\omega, \xi_i) & \geqslant (\tau^{-3} - 1)\mathcal{D}_1 \geqslant \frac{1-\tau}{\tau}\mathcal{D}_1, \text{ and} \\ \inf_i \vartheta(\omega', \eta_i) & \geqslant \mathcal{D}_1', \end{cases}$$

where $\eta_i = \varphi(\xi_i)$ for $i \in \{1, 2, 3\}$. Define $\mathcal{D}_2 = \frac{1-\tau}{\tau}\mathcal{D}_1$. Applying the definition of the metric cross-ratio we deduce from the previous inequalities

$$\left| \log[\omega, \xi_1, \xi_2, \xi_3] - \log \frac{\varrho(\xi_1, \xi_3)}{\varrho(\xi_1, \xi_2)} \right| \leqslant 2|\log \operatorname{diam}(\Xi)| \vee |\log \mathcal{D}_2| \tag{4.10}$$

$$\left| \log[\omega', \eta_1, \eta_2, \eta_3] - \log \frac{\vartheta(\eta_1, \eta_3)}{\vartheta(\eta_1, \eta_2)} \right| \leqslant 2|\log \operatorname{diam}(\Xi)| \vee |\log \mathcal{D}_1'|. \tag{4.11}$$

Denote by $\mathcal{L}$, resp. $\mathcal{L}'$ the right-hand side bounds of (4.10), resp. (4.11). Let $r \in \mathbf{R}_{>0}$ and $\mathfrak{M} \in \mathbf{R}_{\geqslant 0}$ be such that $R = r\exp(\mathfrak{M}) \leqslant \mathcal{D}_1$. Fix ξ_1 and write $B = B(\xi_1, r)$. Fix ξ_2 in $\tau B \setminus \tau^2 B$. For any $\xi_3 \in A = B(R) \setminus B(r)$ the triangle inequality gives

$$\varrho(\xi_1, \xi_2) \wedge \varrho(\xi_1, \xi_3) \wedge \varrho(\xi_2, \xi_3) \geqslant \left((1-\tau) \wedge \tau^2\right) r.$$

Let v_0 be such that φ is $(\underline{\lambda}, \overline{\lambda}, v_0)$-quasiMöbius. Define $v_1 = v_0 + \mathcal{L} \vee \mathcal{L}'$ and then $v_2 = (v_1 \uparrow (1-\tau) \wedge \tau^2) v_1$. Applying Definition 4.1 to φ for $(\omega, \xi_1, \xi_2, \xi_3)$ together with (4.10) and (4.11), one obtains the set of inequalities

$$\begin{aligned} \log \frac{\vartheta(\eta_1, \eta_3)}{\vartheta(\eta_1, \eta_2)} \leqslant \log^+ \frac{\vartheta(\eta_1, \eta_3)}{\vartheta(\eta_1, \eta_2)} &\leqslant \overline{\lambda} \log^+ \frac{\varrho(\xi_1, \xi_3)}{\varrho(\xi_1, \xi_2)} + v_1\left(-\log\left((1-\tau) \wedge \tau^2\right) r\right) \\ &\leqslant \overline{\lambda}\mathfrak{M} + v_2(-\log r) - 2\overline{\lambda}\log\tau, \\ -\log \frac{\vartheta(\eta_1, \eta_3)}{\vartheta(\eta_1, \eta_2)} \leqslant \log^+ \frac{\vartheta(\eta_1, \eta_2)}{\vartheta(\eta_1, \eta_3)} &\leqslant \overline{\lambda} \log \frac{\varrho(\xi_1, \xi_2)}{\varrho(\xi_1, \xi_3)} + v_2(-\log r) = v_2(-\log r). \end{aligned}$$

Hence for any $\xi_3, \xi_3' \in A$, by the triangle inequality in $\mathbf{R}$, using $\vartheta(\eta_1, \eta_2)$ as an intermediate point,

$$\left| \log \frac{\vartheta(\eta_1, \eta_3)}{\vartheta(\eta_1, \eta_3')} \right| \leqslant 2\overline{\lambda}\mathfrak{M} + 2v_2(-\log r) - 4\overline{\lambda}\log\tau \leqslant 2\lambda\mathfrak{M} + w(-\log r), \tag{4.12}$$

where $w = O(u)$. The proposition follows from the last statement. The expansion constant $\underline{\lambda}$ would intervene in lower bounds on inf $\frac{\vartheta(\varphi(\xi_1),\varphi(\xi_3))}{\vartheta(\varphi(\xi_1),\varphi(\xi_2))}$ for ξ_2 in the internal ball, and ξ_3 outside the external ball, centered at ξ_1. □

This last property of sublinearly Möbius maps will be of use in section 5 where we implement some measure theory on the boundary. There is still a need to reformulate it slightly, however, since we will be then working with balls rather than annuli, and quasimetrics rather than true distances. In that purpose, we introduce the following terminology: for any $s \in \mathbf{R}_{>0}$, if B is a quasiball $B = B^{\widehat{\varrho}}(\Xi, r)$ where $\widehat{\varrho}$ is a kernel equivalent to the distance in X, then sB is $B^{\widehat{\varrho}}(\Xi, sr)$. If Ξ is τ-uniformly perfect for every $\tau \in (0,1)$ with respect to $\widehat{\varrho}$ (for instance, if it is connected) this is a continuous operation of $\mathbf{R}_{>0}$ on the space of quasiballs.

Proposition 4.10. *Assume that (Ξ, ϱ) and (Ψ, ϑ) are compact connected topological manifolds, and that $\varphi : \Xi \to \Psi$ is a $(\underline{\lambda}, \overline{\lambda}, O(u))$-sublinearly quasiMöbius homeomorphism. Let $Q \in \mathbf{R}_{\geqslant 1}$ be a constant. Let $\widehat{\varrho}$, resp. $\widehat{\vartheta}$ be an equivalent kernel on Ξ, resp. on Ψ. Then for any $\alpha \in (0, \underline{\lambda})$ and $\beta \in (\overline{\lambda}, +\infty)$ there exists $w = O(u)$ (depending on Q) such that for any $\widehat{\varrho}$-quasiball $B \subset \Xi$ with center ξ and small enough radius r there exists a $\widehat{\vartheta}$-quasiball B' in Ψ, and*

$$\begin{cases} r^{\beta} \leqslant \operatorname{radius}(B') \leqslant r^{\alpha} \\ B' \subseteq \varphi\left(Q^{-1}B\right) \subset \varphi(B) \subseteq Q^{\lambda} e^{w(-\log r)} B'. \end{cases} \tag{4.13}$$

Remark 4.11. Though this would be valid, we do not include in the conclusion that B have center $\varphi(\xi)$, since it will not be required in section 5.

Proof. The statement for any equivalent kernel follows from the particular case when $\widehat{\varrho} = \varrho$ and $\widehat{\vartheta} = \vartheta$. Let $B'' \setminus B'$ be an annulus containing $\varphi(B'' \setminus B')$. Since φ is a homeomorphism, images of balls, resp. spheres by φ are topological balls, resp. spheres. By the Jordan-Brouwer separation theorem, $\varphi(Q^{-1}B)$ is one of the two connected components of $\Psi \setminus \varphi(\partial(Q^{-1}B)$, and by Proposition 4.4, if r is small enough its diameter is bounded by r^{α}. Since $\varphi(\partial(Q^{-1}B)) \subset B'' \setminus B'$, $B' \subseteq \varphi(Q^{-1}B)$ and $\operatorname{radius}(B') \leqslant r^{\alpha}$. By Proposition 4.9 B'' can be written $Q^{\lambda} e^{w}(-\log r) B'$. Finally, by Proposition 4.4, for all $\beta' \in (\overline{\lambda}, \beta)$, $\operatorname{diam}(B'') \geqslant \operatorname{diam} \varphi(B) \geqslant r^{\beta'}$ if r is small enough. This implies the lower bound on $\operatorname{radius}(B')$ for r small enough. □

5. Riemannian negatively curved homogeneous spaces

5.A. **Setting.** Simple Lie groups of real rank one with left invariant metrics are mentioned early in Gromov's essay as important examples of δ-hyperbolic spaces [20, 1.5(2)] and it is natural to ask to which extent they – or their quasiisometrically related symmetric spaces of noncompact type – differ on the large scale. Beyond these examples, it was proved in 1974 by E. Heintze [22, § 2] that any connected homogeneous negatively curved Riemannian manifold is the principal space of a solvable Lie group $S = N \rtimes_{\alpha} \mathbf{R}$, where N is nilpotent with Lie algebra $\mathfrak{n}$ and $\alpha \in \operatorname{Der}(\mathfrak{n})$ is such that for any compact neighborhood K of 1 in N, $\cup_{t \geqslant 0} \exp(t\alpha)K = N$. Such an S is called a Heintze group.

For a principal space X of the Heintze group S, denote by ω the endpoint on $\partial_{\infty} X$ (in positive time) of the orbits of the $\mathbf{R}$ factor, and by $\partial_{\infty}^{*} X$ the punctured boundary $\partial_{\infty} X \setminus \{\omega\}$. Any choice of a basepoint $o \in X$ will determine a chart $\Phi : \partial_{\infty}^{*} X \to N$

by letting $(\omega\xi)$ be the $\Phi(\xi)$-left translate of the $\mathbf{R}$ factor in $(X, o) \simeq (S, 1)$, and a horofunction $-t : X \to \mathbf{R}$ from ω and such that $t(o) = 0$.

5.B. **Quasimetrics and measures on the punctured boundary.** From now on, we make an assumption that S is purely real, i.e. α has only positive real eigenvalues. This is not restrictive as far as large-scale properties are concerned, due to the following fact:

Proposition 5.1. *Any Heintze group is quasiisometric to a purely real Heintze group.*

Proposition 5.1 follows from a special case of a result by D.Alekseevskiĭ [1, Theorem 3.3]. See also Cornulier, [9, Corollary 5.16] for a generalized form.

For any $s \in \mathbf{R}_{>0}$ there is a homomorphism $N \rtimes_{s\alpha} \mathbf{R} \to N \rtimes_\alpha \mathbf{R}$, $(n, t) \mapsto (n, ts)$. Up to rescaling the operation of $\mathbf{R}$, we will work under a normalization assumption:

Definition 5.2. A purely real Heintze group $N \rtimes_\alpha \mathbf{R}$ is normalized if the smallest eigenvalue of α is equal to 1. In this case, the eigenvalues are ordered in increasing order, $1 = \lambda_1, \dots \lambda_r$ and one defines $p = \operatorname{tr} \alpha$.

Lemma 5.3. *Choose a horofunction β from ω in X, and let $\widehat{\varrho}$ be the visual quasimetric on $\partial_\infty^* X$ with parameter e with respect to β. $\widehat{\varrho}$ is a N-invariant, S-equivariant adapted kernel on $\partial_\infty^* X$; precisely*

$$\forall \xi_1, \xi_2 \in \partial_\infty^* X,\ \widehat{\varrho}(s\xi_1, s\xi_2) = e^t \widehat{\varrho}(\xi_1, \xi_2), \tag{5.1}$$

if $s = (n, t)$ in the semidirect product decomposition $N \rtimes \mathbf{R}$.

Proof. Applying s is equivalent to removing t to the horofunction β. □

We refer to $\widehat{\varrho}$ as the homogeneous quasimetric on the punctured boundary; it is indeed a quasimetric (see e.g. Buyalo and Schroeder [5, 3.3]). Different, equally natural choices for $\widehat{\varrho}$ are possible; under the constraint of satisfying (5.1) and a quasiultrametric inequality they would lead to equivalent kernels. We shall give later (Lemma 5.5) a sufficient condition for $\widehat{\varrho}$ to be equivalent to a true distance. For the moment however we only draw measure-theoretic conclusions.

By definition, N operates on $\partial_\infty^* X$, and then on the space of measures on $\partial_\infty^* X$; the invariant subspace is an affine line $\mathcal{L}$, by uniqueness of the Haar measure of N up to scaling. This operation extends to $S \curvearrowright \mathcal{L}$ via its modular function: for any $\mu \in \mathcal{L}$, for any $\widehat{\varrho}$-quasiball B,

$$\forall s \in S,\ \mu(sB) = \Delta(s) \mu(B), \tag{5.2}$$

where $\Delta(s) = \exp(t \cdot \operatorname{tr} \alpha) = e^{pt}$ if $s = (n, t)$, and we recall that p is the trace of α.

5.C. **Horizontal lines and horizontal curves.** Let $\mathfrak{n}_1(\alpha) = \ker(\alpha - 1)$. In the tangent space of $\partial_\infty^* X$, the distribution $\Phi^* \mathfrak{n}_1(\alpha)$ does not depend on the chart $\Phi : \partial_\infty^* X \to N$. We refer to it as the horizontal distribution, and denote it by τ (not forgetting the left action of N). For any N-invariant line L in τ, denote by Γ_L the family of horizontal L-lines in $\partial_\infty^* X$, that is, smooth horizontal curves γ tangent to L. The space Γ_L can be identified with the homogeneous space N/R, where R is a one-parameter subgroup of N whose infinitesimal generator represents $\Phi_* L$. Since N is a nilpotent Lie group it is unimodular, especially Δ_N is constant along R so that Γ_L possesses a Haar measure ρ, following A.Weil [29, § 9].

Lemma 5.4. *Let L be as above, and let μ be a N-invariant measure on $\partial^*_\infty X$. Then for any $Q \in \mathbf{R}_{>0}$ there exists $c \in \mathbf{R}_{>0}$ (depending on μ, Q and L) such that for any $\widehat{\varrho}$-quasiball B,*

$$\rho\left\{\gamma \in \Gamma_L : \gamma \cap B \neq \emptyset\right\} = c\mu\left(Q^{-1}B\right)^{(p-1)/p}. \tag{5.3}$$

Proof. S operates simply transitively on the space of $\widehat{\varrho}$-quasiballs, while N operates simply transitively on their centers preserving radii, and $\theta : B \mapsto \{\gamma : \gamma \cap B \neq \emptyset\}$ defines a S-equivariant map. Hence it suffices to show (5.3) for a one-parameter family of balls $\{e^t B\}_{t\in\mathbf{R}}$. Let $v \in \mathfrak{n}_1(\alpha)$ be a nonzero vector such that $[\Phi^* v] = L$. Since $v \in \mathfrak{n}_1(\alpha)$, the linear map α operates on $\mathfrak{n}/(\mathbf{R}v)$ with trace $p-1$, and $\rho\left\{\gamma \in \Gamma_L : \gamma \cap e^t B \neq \emptyset\right\}$ is proportional to $e^{t(p-1)}$. On the other hand, by Lemma 5.3 and (5.2), $\mu(Q^{-1}tB)$ is proportional to e^{tp}, hence $\mu(Q^{-1}tB)^{(p-1)/p}$ is proportional to $e^{t(p-1)}$ as well. □

Lemma 5.5. *Assume that S is normalized (Definition 5.2) and that the operation of the derivation α on $\mathfrak{n}^{\mathrm{ab}}$ is scalar, hence the identity. Then*

1. *(N, α) is a Carnot graded group, i.e.*
   (a) *$\mathfrak{n}$ admits a grading $(\mathfrak{n}_i)$ by $\mathbf{Z}_{>0}$ such that $\mathfrak{n}_i = \ker(\alpha - i)$*
   (b) *$\mathfrak{n}$ is generated by $\mathfrak{n}_1$.*
2. *Let $\|\cdot\|$ be a norm on $\mathfrak{n}_1$, and let $\Phi : \partial^*_\infty X \to N$ be a chart. Then $\widehat{\varrho}$ is equivalent to a subRiemannian Carnot-Caratheodory metric*

   $$d_{\mathrm{CC}}(\xi_1, \xi_2) = \inf\left\{\ell(\gamma) : \gamma \in \Gamma(\xi_1, \xi_2)\right\},$$

   where $\Gamma(\xi_1, \xi_2)$ denotes the space of absolutely continuous curves $[0,1] \to \partial^*_\infty X$ between ξ_1 and ξ_2 with derivative almost everywhere in the horizontal distribution τ, and $\ell(\gamma) = \int_{[0,1]} \|\Phi_* \gamma'\|$ is the length of γ.

In this case, X is said to be of Carnot type (following Cornulier's terminology).

If X is of Carnot type, condition (1b) ensures that $\Gamma(\xi_1, \xi_2)$ is never empty and d_{CC} takes finite values.

Proof. See the survey of Cornulier [10, 2.G.1 and 2.G.2] for (1). Further, $s = (n,t) \in S$ acts on the space of horizontal curves sending $\Gamma(\xi_1, \xi_2)$ on $\Gamma(s\xi_1, s\xi_2)$ and multiplying lengths by e^t, hence

$$d_{\mathrm{CC}}(s\xi_1, s\xi_2) = e^t d_{\mathrm{CC}}(\xi_1, \xi_2) \tag{5.4}$$

for all $\xi_1, \xi_2 \in \partial^*_\infty X$. Select $\xi \in \partial^*_\infty X$. Since d_{CC} and $\widehat{\varrho}$ are both quasimetrics, they are continuous, hence bounded, over unit quasiballs of each other centered at ξ. Finally, S operates transitively on the spaces of quasiballs of $\widehat{\varrho}$ and d_{CC}. Hence $\widehat{\varrho}$ and d_{CC} are equivalent (the control constants depend on $\|\cdot\|$). □

5.D. **Volumes of quasiballs and intersecting horizontal lines.**

Lemma 5.6. *Let $q \in \mathbf{R}_{\geqslant 1}$ be a constant and let $\mathcal{X}$ be a proper metric space. Let $\widehat{\varrho}$ be an equivalent kernel on $\mathcal{X}$ with quasi-ultrametric constant q. There exists a constant Q depending on q, such that for any countable covering $\mathcal{B}$ of $\mathcal{X}$ by $\widehat{\varrho}$-quasiballs, there exists an extraction $\mathcal{B}'$ of $\mathcal{B}$ whose elements are disjoint and such that $\{QB\}_{B\in\mathcal{B}'}$ is a covering of X.*

Proof. See A.P. Morse, [24, Theorem 3.4], or Federer [17, 2.8.4-2.8.6]. □

In the following, whenever $q \in \mathbf{R}_{\geqslant 1}$ is a constant, Q is another constant depending on q defined by the previous lemma.

Lemma 5.7 (adapted from P.Pansu, [26, Lemme 6.3]). *Let $\mathcal{X}$ be a proper metric space, and let Γ a measured space of curves on $\mathcal{X}$ (denote its measure by ρ). Let $p \in \mathbf{R}_{>1}$ and $q \in \mathbf{R}_{\geqslant 1}$ be constants. Let $\widehat{\varrho}$ be a kernel on $\mathcal{X}$, equivalent to the original distance and with a q-quasiultrametric inequality. Let U be an open, bounded subset of $\mathcal{X}$, endowed with Borel measures μ and ν, such that for any $\widehat{\varrho}$-quasiball B contained in U,*

$$\rho\left\{\gamma \in \Gamma : \gamma \cap B \neq \emptyset\right\} \leqslant \mu\left(Q^{-1}B\right)^{(p-1)/p}. \tag{H}$$

For all $\gamma \in \Gamma$ and for all $r > 0$, set

$$\Phi_r^1(\gamma) = \inf_{\mathcal{F}} \sum_{B \in \mathcal{F}} \phi(B),$$

where $\phi(B) := \nu\left(Q^{-1}B\right)^{1/p}$, the infimum taken over countable coverings $\mathcal{F}$ of $\gamma \cap U$ with balls of radius r exactly, contained in U. Then

$$\int_\Gamma \Phi_r^1(\gamma) d\rho \leqslant \nu(U)^{1/p}\mu(U)^{(p-1)/p}. \tag{5.5}$$

For Lemma 5.7, Pansu's proof can be reproduced almost verbatim [26, Lemme 6.3], with the only differences of using Lemma 5.6 instead of the covering lemma used by Pansu, having r fixed and not going to the limit at the end. The argument is based on the Hölder inequality; in a more general setting it is aimed at bounding a discretized version of the conformal modulus, and then to obtain lower bounds for the conformal dimension, [25, § 2 and 3].

Lemma 5.8 (compare [26, Proposition 6.5]). *Let (N, α) and (N', α') be Carnot groups with grading derivations α, α', normalized, with positive eigenvalues, of traces p and p'. Let X and X' be principal spaces of $N \rtimes_\alpha \mathbf{R}$ and $N' \rtimes_{\alpha'} \mathbf{R}$ respectively, and assume there exists a homeomorphism $\partial_\infty^* X \to \partial_\infty^* X'$ which is sublinearly quasiMöbius over every compact subset. Then $p \leqslant p'$.*

Proof sketch. Define τ as p'/p and let Γ_L be a family of horizontal lines in the boundary of X. We follow the lines of Pansu [26, Proposition 6.5], despite loosing strength in the conclusion. Precisely this amounts to comparing two facts:

(1) Without any assumption on N and α, for any $\sigma \in (\tau, +\infty)$, the image of almost every horizontal curve $\gamma \in \Gamma_L$ has locally finite σ-dimensional $\widehat{\varrho}$-Hausdorff measure. Hence almost every curve has $\widehat{\varrho}$-Hausdorff dimension less than τ.
(2) Since X is of Carnot type, $\widehat{\varrho}$ is equivalent to the subRiemannian distance d_{CC} by Lemma 5.5, hence any nonconstant curve should have $\widehat{\varrho}$-Hausdorff-dimension greater than 1.

This proves that $\tau \geqslant 1$, i.e. $p \leqslant p'$.

Proof. Let U be a open, relatively compact subset of $\partial_\infty^* X$. Define $U' = \varphi(U)$. Let Γ_L^U be the (non-empty) set $\{\gamma \cap U : \gamma \in \Gamma_L\}$ measured with

$$(\cap_U)_\star \left(\rho \lfloor \{\gamma \in \Gamma_L : \gamma \cap U \neq \emptyset\}\right),$$

where ρ has been defined in 5.B, and $\cap_U(\gamma) = U \cap \gamma$. We still denote this measure ρ. Let μ, resp. μ' be a N-invariant measure on $\partial_\infty^* X$, resp. on $\partial_\infty^* X'$, restricted to U, resp. to U'. Define a measure ν on U as

$$\nu(B) = \mu'(\varphi(B))$$

for any Borel subset $B \subset U$. Let $\widehat{\varrho}$ be the homogeneous quasimetric on $\partial_\infty^* X$, let q be its ultrametric constant and define Q accordingly (see Lemma 5.6). Let $r \in \mathbf{R}_{>0}$ be a radius that will be repeatedly assumed as small as needed. Choose $\gamma \in \Gamma_L^U$, and let $\mathcal{F}$ be any covering of γ with quasiballs of the same $\widehat{\varrho}$-radius r (we emphasize that all quasiballs must have radius r). By assumption, the quasiballs $\{\varphi(B), B \in \mathcal{F}\}$ cover $\varphi(\gamma)$. By Theorem 4.2 and Proposition 4.10, there exists $v = O(u)$, and if r is small enough, a collection $\mathcal{F}'$ of quasiballs and $\mathcal{F} \to \mathcal{F}'$, $B \mapsto B'$ such that

$$\forall B \in \mathcal{F},\, B' \subset \varphi\left(Q^{-1} B\right) \subset \varphi(B) \subset Q^{2\lambda} e^{v(-\log r)} B' =: B''. \tag{5.6}$$

Define $\mathcal{F}'' = \{B''\}$ together with a map $\mathcal{F} \to \mathcal{F}'', B \mapsto B''$. This is a quasiball covering of $\varphi(\gamma)$.

Next, define a gauge function $\phi(B) := \nu\left(Q^{-1}B\right)^{1/p} = \mu'\left(\varphi\left(Q^{-1}B\right)\right)^{1/p}$. There exists a constant $c_0 \in \mathbf{R}_{>0}$, not depending on r and such that

$$\begin{aligned} \phi(B) &\overset{(5.6)}{\geqslant} \mu'(B')^{1/p} = c_0^\tau \operatorname{diam}(B')^\tau \\ &\overset{(5.6)}{\geqslant} \left(\frac{c_0}{Q^{2\lambda} e^{v(-\log r)}}\right)^\tau \operatorname{diam}(B'')^\tau. \end{aligned} \tag{5.7}$$

Define $r'' = r^{1/(2\lambda)}$. Using Cornulier's theorem 1.7, if r is small enough, then

$$\begin{aligned} \forall B \in \mathcal{F},\, \operatorname{diam} B'' \leqslant e^{v(-\log r)} Q^{2\lambda} \operatorname{diam} B' &\leqslant e^{v(-\log r)} Q^{2\lambda} \operatorname{diam} \varphi(B) \\ &\leqslant e^{v(-\log r)} Q^{2\lambda} r^{2/(3\lambda)} \\ &\leqslant r'', \end{aligned}$$

where we used $v(s) \ll s$ and took r small enough in the last line. On the other hand, using (4.8) from the proof of Proposition 4.4, one obtains a reverse inequality:

$$\forall B'' \in \mathcal{F},\, \log \operatorname{diam} B'' \geqslant 2\lambda \log r = 4\lambda^2 \log r''. \tag{5.8}$$

One can rewrite $Q^{2\lambda} e^{v(-\log r)}$ as $e^{w(-\log r'')}$ with $w = O(u)$. Taking logarithms in (5.7),

$$\begin{aligned} \log \phi(B) &\geqslant \tau \log c_0 - w(-\log r'') + \tau \log \operatorname{diam} B'' \\ &\underset{(5.8)}{\geqslant} \tau \log c_0 - w\left(-\frac{1}{4\lambda^2} \log \operatorname{diam} B''\right) + \tau \log \operatorname{diam} B''. \end{aligned}$$

The function w is strictly sublinear, so for any $\sigma \in (\tau, +\infty)$, there is $r_\sigma \in \mathbf{R}_{>0}$ such that

$$\forall r \in (0, r_\sigma), \forall B \in \mathcal{F},\, \phi(B) \geqslant (r'')^\sigma \geqslant (\operatorname{diam} B'')^\sigma. \tag{5.9}$$

Recall that for all $\mathcal{F}$ the quasiballs $B'' \in \mathcal{F}''$ cover $\varphi(\gamma)$. By definition of the $\widehat{\varrho}$-Hausdorff premeasure at scale r'',

$$\Phi_r^1(\gamma) = \inf_{\mathcal{F}} \sum_{B \in \mathcal{F}} \phi(B) \geqslant \sum_{B''} \operatorname{diam}(B'')^\gamma \geqslant \mathscr{H}_{r''}^\sigma \varphi(\gamma). \tag{5.10}$$

By Lemma 5.4, the hypothesis (H) of Lemma 5.7 is fullfilled. Hence, for all $r \in (0, r_\sigma)$,

$$\int_{\Gamma_L^U} \Phi_r^1(\gamma) d\rho \leqslant \nu(U)^{1/p} \mu(U)^{(p-1)/p}.$$

By monotone convergence, for ρ-almost every γ, $\sup_r \Phi_r^1(\gamma)$ is finite, and then by (5.10), $\mathscr{H}^\sigma \varphi(\gamma)$ is finite. Considering this fact for all terms of a decreasing sequence $\{\sigma_j\}$ converging to τ, one deduces that, still for ρ-almost every γ,

$$\dim_{\mathrm{H}} \varphi(\gamma) \leqslant \inf_j \sigma_j = \tau. \tag{5.11}$$

Finally, X has been assumed of Carnot type, hence $\widehat{\varrho}$ is equivalent to the Carnot-Caratheodory metric d_{CC} by Lemma 5.5. By the triangle inequality, the 1-dimensional d_{CC}-Hausdorff measure of any nonconstant curve is nonzero, in particular its d_{CC}-Hausdorff dimension must be greater than 1. This dimension does not change when replacing d_{CC} with the equivalent quasimetric $\widehat{\varrho}$. By (5.11) there exists $\gamma \in \Gamma_L^U$ such that $1 \leqslant \dim_{\mathrm{H}} \varphi(\gamma) \leqslant \tau$. Hence $1 \leqslant \tau$. □

Lemma 5.8 is applied to show that p is a SBE invariant between spaces of Carnot type. In fact this can be made slightly more general:

Proposition 5.9. *Let X_1 and X_2 be principal spaces of purely real, normalized Heintze groups $N_1 \rtimes_{\alpha_1} \mathbf{R}$ and $N_2 \rtimes_{\alpha_2} \mathbf{R}$. Assume that for all $i \in \{1,2\}$ the operation defined by α_i on $\mathfrak{n}_i^{\mathrm{ab}}$ is unipotent. If there exists a sublinearly biLipschitz equivalence between X_1 and X_2, then $\mathrm{tr}(\alpha_1) = \mathrm{tr}(\alpha_2)$.*

Proof. For every $i \in \{1,2\}$, decompose α_i into $\alpha_i^\sigma + \alpha_i^\nu$, where α_i^σ is semi-simple and α_i^ν is nilpotent. By hypothesis, $\mathfrak{n}_i^\sigma$ operates as the identity on $\mathfrak{n}_i^{\mathrm{ab}}$, hence N_i are Carnot gradable groups, and α_i are grading derivations of their Lie algebra. A particular instance of a theorem by Cornulier implies that there exists $O(\log)$-sublinearly biLipschitz equivalences $\psi_i : N \rtimes_{\alpha_i^\sigma} \mathbf{R} \to N \rtimes_{\alpha_i} \mathbf{R}$ (see [8, Theorem 4.4]: in our very special case the exponential radical is N, and the Cartan subgroup is $\mathbf{R}$). The groups $N \rtimes_{\alpha_i^\sigma} \mathbf{R}$ are of Carnot type, so by Theorem 4.2 and Lemma 5.8, $\mathrm{tr}(\alpha_1^\sigma) = \mathrm{tr}(\alpha_2^\sigma)$. Finally, $\mathrm{tr}(\alpha_1) = \mathrm{tr}(\alpha_1^\sigma) = \mathrm{tr}(\alpha_2^\sigma) = \mathrm{tr}(\alpha_2)$. □

Note that if sublinearly biLipschitz equivalences are replaced by quasiisometries in the last statement, known invariants are much finer than the trace. In this direction, M. Carrasco Piaggio and E. Sequeira obtained that for normalized purely real Heintze groups, resp. for normalized purely real Heintze groups with a fixed Heisenberg group as exponential radical N, the characteristic polynomial, resp. the full Jordan form of α, are quasiisometric invariants [6, Theorem 1.1, resp. Theorem 1.3]. By contrast, the Jordan form of the normalized derivation is not a SBE invariant, precisely it is not a $O(\log)$-SBE invariant by Cornulier's theorem [8, Theorem 4.4].

5.E. **Proof of Theorem 2.** Notation is as before. When X is a rank one symmetric space of noncompact type, several restrictions appear (see Heintze, [22, Proposition 4 and Corollary]):

(1) X is of Carnot type.
(2) The Lie algebra $\mathfrak{n}$ is two-step, $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ where $\mathfrak{n}_2$ is possibly zero.

(3) Save for one case, namely the Cayley hyperbolic plane, there exists a division algebra structure on $\mathbf{R} \oplus \mathfrak{n}_2$, and $\mathfrak{n}_1$ is a module over this division algebra. The structure of $\mathfrak{n}$ is completely determined by these data.

The Frobenius classification of division algebras over $\mathbf{R}$ reduces considerably the list of candidates thanks to (3): the two relevant parameters are the division algebra $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$ and a positive integer, the rank of $\mathfrak{n}_1$ over $\mathbf{K}$. The Cayley hyperbolic plane fits in this list, setting $\mathfrak{n}_1 = \mathbf{O}$. The homogeneous dimension is computed as

$$\begin{aligned}\operatorname{tr}(\alpha) = \dim \mathfrak{n}_1 + 2\dim \mathfrak{n}_2 &= \dim \mathfrak{n} + \dim \mathfrak{n}_2 \\ &= \dim X - 1 + \dim \mathfrak{Im}(\mathbf{K}),\end{aligned}$$

and $\mathbf{K}$ is completely determined by $\dim \mathfrak{Im}(\mathbf{K}) \in \{0, 1, 3, 7\}$. By Theorem 1.7 and Proposition 5.9, $\mathbf{K}$ is a SBE invariant, as

$$\dim \mathfrak{Im}(\mathbf{K}) = \dim_{\mathrm{H}}(\partial_\infty^* X, \widehat{\varrho}) - \dim \partial_\infty X.$$

The rank n of $\mathfrak{n}_1$ over $\mathbf{K}$ is a SBE invariant as well, since it can be computed by the formula

$$(1+n)\dim_{\mathbf{R}} \mathbf{K} = 1 + \dim \partial_\infty X.$$

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Laboratoire de Mathématiques d'Orsay, Univ. Paris-Sud, CNRS, Université Paris-Saclay, 91405 Orsay, France.

E-mail address: gabriel.pallier@math.u-psud.fr