Large-scale sublinearly Lipschitz hyperbolic geometry

Gabriel Pallier
Université Paris-Sud
gabriel.pallier@math.u-psud.fr

MATHEMATIQUES ORSAY

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Abstract

Large-scale sublinearly Lipschitz maps have been introduced by Yves Cornulier as a precise way to state his results on asymptotic cones of Lie groups; those generalize quasiisometries. Cornulier asks about the effects of those maps on other asymptotic invariants [2]. We focus here on the boundaries of hyperbolic spaces and exhibit an almost quasiconformal behaviour. In favorable situations this still allows some analysis at the boundary.

Introduction: $\mathbb{H}^3_{\mathbf{R}}$ and its boundary

Let $\mathbf{S}$ be the Riemann sphere, g a constant curvature metric on $\mathbf{S}$ with length metric d. A positive homeomorphism $\varphi : \mathbf{S} \to \mathbf{S}$ is an automorphism if one of the following holds:

i. φ is conformal with respect to d, i.e. sends circles of d on circles of d;
ii. φ is conformal with respect to g, i.e. differentiable, preserving infinitesimal circles;
iii. φ preserves the norm of the cross-ratio, defined in terms of distances in an affine chart by

$$[\zeta_1, \zeta_2; \zeta_3, \zeta_4] = \frac{|\zeta_3 - \zeta_1|}{|\zeta_4 - \zeta_1|} : \frac{|\zeta_3 - \zeta_2|}{|\zeta_4 - \zeta_2|}.$$

If $\mathbf{S}$ is at the boundary of real hyperbolic 3-space (this is natural, for instance in the projective model), totally geodesic planes of $\mathbb{H}^3_{\mathbf{R}}$ are bounded by real projective lines, i.e. circles of d, and distances within $\mathbb{H}^3_{\mathbf{R}}$ can be expressed in terms of metric cross-ratios, see below and [4].

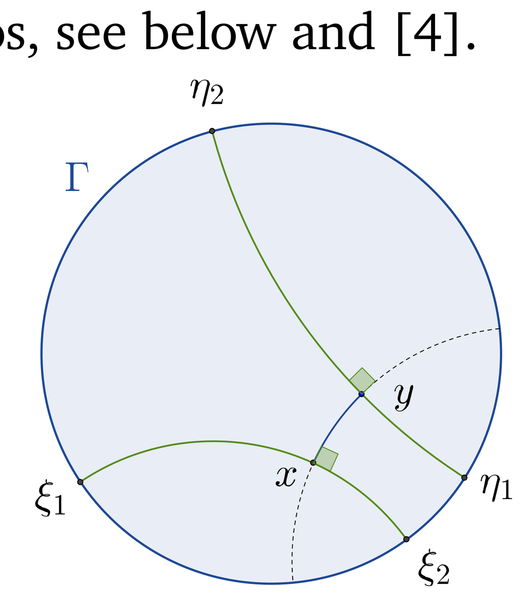

Figure 1: x and y in $\mathbb{H}^3_{\mathbf{R}}$; a totally geodesic plane with boundary $\Gamma \simeq \mathbf{R}$, containing x and y. Geodesics in green. The distance between x and y is *up to an additive bounded error*, $\log^+[\eta_2, \xi_1; \xi_2, \eta_1]$ where $\log^+(s) := \max(0, \log s)$.

The action α of $\mathrm{Isom}^+(\mathbb{H}^3_{\mathbf{R}})$ on the boundary $\partial\mathbb{H}^3_{\mathbf{R}}$ can thus be described by two words : conformal, or Möbius (that is, preserving the metric cross-ratio). α is faithful, reaches the full conformal group of the boundary, and characterization ii says that it is smooth, hence more regular than expected.

This interaction between hyperbolic and conformal/Möbius geometry has been vastly investigated since the 1960s, in the coarser setting of quasiisometries and quasiconformal geometry at the boundary. It is instrumental in proofs of rank one Mostow rigidity, as well as Sullivan and Tukia's theorems (for modern accounts, see [1, 3]).

Our aim here is to quasify further in order to obtain information about Cornulier's sublinearly Lipschitz maps between Gromov-hyperbolic metric spaces.

Sublinearly Lipschitz maps

Let X and Y be pointed metric spaces ; denote the distances to the base-points by $|\cdot|$. A map $f : X \to Y$ is a large-scale sublinearly biLipschitz equivalence (SBE) if there exists $u : \mathbf{R}_{\geqslant 0} \to \mathbf{R}_{\geqslant 1}$ such that $u(r) \ll r$ and constants, $\lambda, \overline{\lambda} \in \mathbf{R}_{>0}$ such that for all $x, x' \in X$ and $y \in Y$,

- $\lambda d(x, x') - u(|x| + |y|) \leqslant d(f(x), f(x') \leqslant \overline{\lambda} d(x, x') + u(|x| + |x'|)$, and
- $d(y, f(X)) \leqslant u(|y|)$.

Setting $u = O(1)$, one recovers quasiisometric maps. As proved by Cornulier, $\mathbb{H}^3_{\mathbf{R}}$ and its deformation $\mathbf{R}^2 \rtimes_N \mathbf{R}$, where N is a unipotent, non-identity matrix, are $O(\log^+)$-SBE yet not quasi-isometric.

Current results

Cornulier proves [2, Theorem 4.4] that a $O(u)$-SBE map $f : X \to Y$ between proper geodesic hyperbolic spaces, induces $\varphi = \partial_\infty f : \partial_\infty X \to \partial_\infty Y$, a biHölder homeomorphism for visual metrics.

Theorem Under the same assumptions, $\partial_\infty f$ is $O(u)$-almost quasiconformal.

Under some hypothesis on X (e.g. nonelementary hyperbolic group), one recovers faithfulness: if $f, g : X \to X$ are SBE maps such that $\partial_\infty f = \partial_\infty g$, then $|f(x) - g(x)| = o(|x|)$.

Proposition Any biHölder, almost quasiconformal homeomorphism between open subsets of Carnot groups preserves the Hausdorff dimension.

Hyperbolic symmetric spaces

Metrically, the Riemannian symmetric spaces of the noncompact type are CAT(0), and hyperbolic when of rank one. Here is the list of the latter:

$$X = \mathbb{H}^n_{\mathbf{R}}, \mathbb{H}^n_{\mathbf{C}}, \mathbb{H}^n_{\mathbf{H}} (n \geqslant 2), \mathbb{H}^2_{\mathbf{O}}. \qquad (1)$$

Maximal unipotent subgroups of $\mathrm{Isom}(X)$ are Carnot groups; with Carnot-Caratheodory (CC) metrics, those provide conformal charts for $\partial_\infty X$ (see figure 3). The list is short enough to allow classification by the combined Lebesgue (topological) and Hausdorff dimensions of the boundaries.

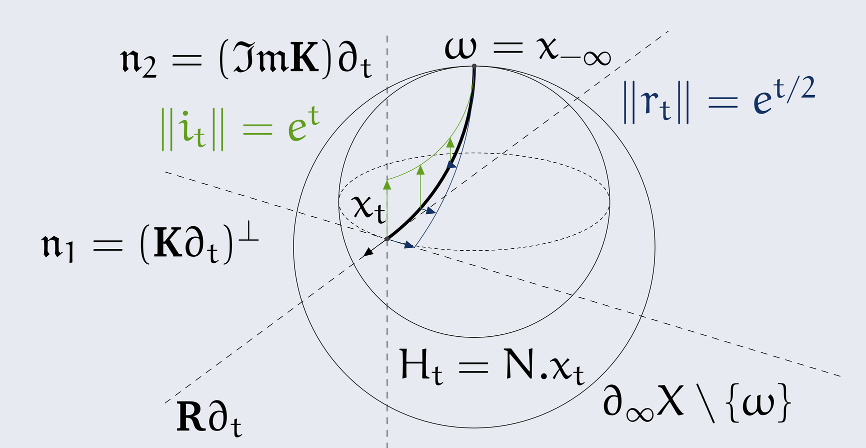

Figure 3: Maximal unipotent N with limit point ω, horofunction t. Along geodesic (x_t), Jacobi fields r_t tangent to a $\mathbf{R}$-plane of curvature $-1/4$ and i_t tangent to a $\mathbf{K}$-line of curvature -1.

Almost quasiconformality

The following introduces some new terminology:

Definitions Let Ξ and Ψ be (quasi)metric spaces, u as above. A subset D of Ξ or Ψ is a (r, t)-annulus if there exists ξ such that $D \subseteq B(\xi, rt) \setminus B(\xi, r)$. Further, say that a homeomorphism $\varphi : \Xi \to \Lambda$ is

- $O(u)$-*almost quasiconformal* if any (r, t)-annulus is sent on a (r', t')-annulus of Λ, where
$$\ln t' = O(\log t) + O(u(-\log r)),$$
- $O(u)$-*almost Hölder-quasiconformal* if there exists $\gamma \in \mathbf{R}_{>0}$ such that one can choose $\ln r' = \gamma \ln r$ in the previous condition.
- $O(u)$-*almost quasiMöbius* if for distincts $\xi_i \in \Xi$,
$$\log^+[\varphi(\xi_1) \cdots \varphi(\xi_4)] = O(\log^+[\xi_1, \xi_2; \xi_3, \xi_4]) + O(u(-\inf \log |\xi_i - \xi_j|)),$$
where $[\xi_1, \dots \xi_4]$ is the metric cross-ratio, and $|\xi_i - \xi_j|$ the distance between ξ_i and ξ_j in Ξ.

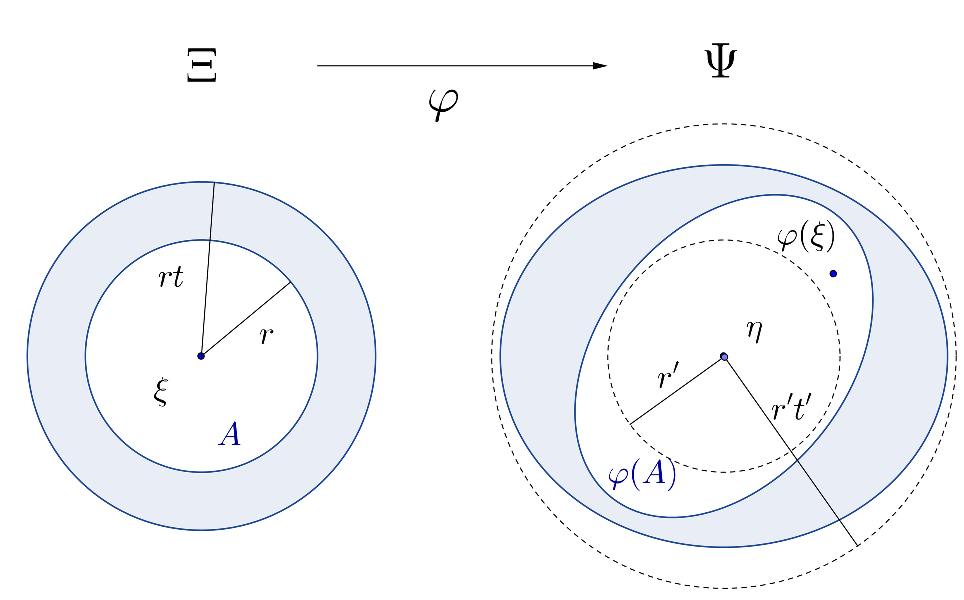

Figure 2: Almost quasiconformal map. Ξ and Ψ must be thought of as quasiconformal charts for boundaries of hyperbolic spaces.

Corollary (question of Druţu [2, 1.16(2)]) $\mathbb{H}^4_{\mathbf{R}}$ and $\mathbb{H}^2_{\mathbf{C}}$ (as well as other pairs of distinct hyperbolic symmetric spaces) are not SBE.

Indeed, maximal unipotent subgroups for $\mathbb{H}^4_{\mathbf{R}}$ and $\mathbb{H}^2_{\mathbf{C}}$ are resp. $\mathbf{R}^3$ and the first Heisenberg group, which has topological dimension 3 but Hausdorff dimension 4 once equipped with a CC metric.

Open questions

- We expect boundary maps to be almost quasiMöbius, and under additional hypotheses almost Hölder-quasiconformal.
- (Fullness) Is any almost Hölder-quasiconformal $\varphi : \partial_\infty X \to \partial_\infty Y$ induced by a SBE map?
- Is there no $u = o(\log^+)$ such that (for instance) $\mathbf{R}^2 \rtimes_N \mathbf{R}$ and $\mathbb{H}^3_{\mathbf{R}}$ are $O(u)$-SBE?

Bibliography

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